SOME TWO-DIMENSIONAL ELASTIC INCLUSION PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled 'SOME TWO-DIMENSIONAL ELASTIC INCLUSION PROBLEMS' that is being submitted by Shri C. B. Sharma, M. Sc., for the award of the Degree of Doctor of Philosophy to the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standard fulfilling the requirements of the regulations to the Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

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SOME TWO-DIMENSIONAL ELASTIC INCLUSION PROBLEMS

This thesis is concerned with a class of inclusion problems in two-dimensional elasticity. 'Inclusion' has been defined as a region, having the same elastic properties as that of the surrounding material, the 'matrix'. Inclusion tends to undergo spontaneous deformation. This tendency would result in prescribed strains in the inclusion, in the absence of the matrix. However, because of the constraints of the matrix, a system of elastic field develops both in the matrix and in the inclusion. Such problems have been studied in this thesis.

The complex-variable method has been applied to solve these problems. The results depend upon the knowledge of the effect of point-force. When the point-force acts upon an infinite medium, the results are well known and can be found practically in all important works on elasticity. These results have been used to find the solution when the circular inclusion tends to undergo any general type of spontaneous deformation. (Previous workers had considered only a uniform strain). This generalisation gives results,

which have important physical interpretations. Such generalization is possible even for the elliptic inclusion problem, but only a particular example has been solved to illustrate the basic ideas. As will be obvious, these results can be applied for inclusions of various shapes. A converse problem can also be tackled, namely what happens if the matrix, in place of inclusion, undergoes spontaneous deformation. One such problem is solved in this thesis but the results can be generalised.

For a semi-infinite region or an infinite strip when the leading edges are free from stresses or displacements, the results of the effect of a point-force in the interior are not readily available. Moreover, some known results for a half plane can be applied only after considerable manipulations. In this thesis, however, the results are given which enable to find exact analytical solution to the circular inclusion problem when it is embedded in a semi-infinite region or is symmetrically situated in an infinite strip. In the latter case of an infinite strip the results are obtained in terms of an infinite integral which have been solved numerically and results are given in tabulated form. In both the cases the problem has been solved for the two cases namely when the leading edges are free from tractions and displacements.

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INTRODUCTION

This thesis concerns itself with a class of inclusion problems in the infinitesimal theory of elasticity. The problem may be stated briefly as follows :

A limited region in an isotropic elastic medium tends to undergo spontaneous dimension changes. If the elastic properties of limited region are the same as those of surrounding material, the 'matrix', it will be termed as 'inclusion' otherwise it would be called 'inhomogeneity'. This spontaneous deformation would be a prescribed strain in the absence of elastic constraints of the matrix. The mutual constraints of the inclusion and the matrix generate a system of locked up accommodation stresses in both the regions. The problem is to find the elastic field and consider related problems in the inclusion and matrix.

The problem is not only a mathematical one but has important applications. For example, such problems do arise in various investigations of physics and technology, e.g. in brittle fracture, precipitation hardening, alloy cohesion, restricted plastic flow.

For an important application, reference may be made to the physical observation made by Hale and Mclean ((45))*. It was subsequently explained on the basis of above mathematical model in ((46)) by Bhargava and Mclean.

They are of great theoretical interest also, as we shall see that on the equilibrium interface the elastic displacements of inclusion and matrix are not continuous whilst the net displacement, made up of elastic and non-elastic contribution, is everywhere continuous. Expressed in a different language one is concerned with states of elastic strain which do not satisfy the compatibility relations of Saint-Venant and which are, nevertheless, realized without material being ruptured.

The simple problem of spherical inclusion in an infinite isotropic elastic continuum was examined by Frenkel ((31)) in connection with his kinetic theory of liquids, and by Mott and Nabarro ((9)) and Nabarro ((32)) in connection with their theory of precipitation hardening in alloys. A systematic investigation of the ellipsoidal inclusion was undertaken in 1957

*Figures enclosed in such double parenthesis refer to the numbers in the bibliography on page 152.

by Eshelby ((10)), where he made use of what may be described as point-force technique.

Even though three-dimensional inclusion problems are more realistic, they involve analytically intractable integrals of formidable nature. But the problem is comparatively simpler in two-dimensional situations as in the cases of plane strain or plane stress problems. It is because the complex-variable method can be applied. This method was formulated by Jaswon and Bhargava ((13)). They illustrated the method by solving the elliptic inclusion problem.

Another method of solving such problems was given by Bhargava ((44)). This was the application of classical minimum energy principle to solve such problems. It was applied by Bhargava and Radhakrishna ((17, 18)) to solve the problem of an elliptic inhomogeneity in an isotropic medium. This method was subsequently applied by them to solve a more general problem, when the inhomogeneity and matrix were of different orthotropic material. Willis ((33)) gave the solution of a simpler problem of an elliptic inclusion in a cubic material, by point-force technique.

Substantial contributions, to two-dimensional elastic inclusion problems, were made by Kapoor ((27)). He not only dealt with the problem of inclusion of various shapes e.g. rectangular and triangular but also solved the problems where the inclusion interacted with an inclusion or an inhomogeneity or a cavity in an otherwise infinite medium.

Recently R. J. Knops ((34)) derived an equation for the strains of an arbitrary elastic field in an infinite matrix perturbed by several inclusions and solved it exactly when the shear moduli of inhomogeneities and matrix are identical and when only a single ellipsoidal inclusion perturbs a field uniform at infinity. Some other recent contributions in this field of study have been concerned with using variational methods to derive bounds for the aggregate moduli of multiphased materials having arbitrary phase geometry. In this connection, reference may be made to the work of Hashin and Shrikman ((35)) and Hill ((36, 37, 38)) where, in particular, bounds are presented for the bulk and shear moduli. Hill ((39)) also estimated the overall moduli of an arbitrary fibre composite with transversely isotropic phases and also the macroscopic elastic moduli of two phase composites ((40)). Budiansky ((41)) gave

an analysis for the determination of the elastic moduli of a composite material. The bounds for elastic moduli of solid composite materials were given by Walpole ((42, 43)) by employing extremum-principles.

The present work is concerned with the extension of such problems. It appears necessary to remark at this stage, that the previous works confined to the case when the spontaneous deformation is characterised by a simple relation i.e. when the strain components are constants. Now if we take the spontaneous deformation characterized by more general relations, this certainly makes the analysis involved but in turn the solution is more general. This thesis deals with a class of such problems.

In passing we have dealt with another aspect of the problem when the matrix tries to undergo spontaneous deformation. This has been discussed in chapter V. Although more general problems can be solved, but technique used in this chapter can be directly applied.

An important aspect in the solution of inclusion problems is the extent of the matrix. In most of the previous works on this subject, the matrix was supposed to be infinite in all directions. A first step in reducing the size of the matrix is to consider it

semi-infinite. It seems to have important applications in engineering and technology. Damage to the structural frames resulting from swelling of clay soils used as foundations has been well documented over years. In most of the cases the damage has been attributed to vertical component of swelling and also to the horizontal component. The simple model of homogeneous isotropic elastic material which has been assumed in the present work may be a simplification of the soil mass which is an inelastic continuum, but it may be a first step towards solution of above problems.

Another step is to consider the matrix as a medium consisting of an infinite elastic strip. Solution of inclusion problems in infinite and semi-infinite media becomes particular cases of such a solution. Problem of an inclusion in an infinite strip has also been considered in this thesis.

We recount below the work done in this thesis. In the first two chapters, we have given relevant theory of complex variable approach and the point-force technique, which has all through been used. In chapter III the problem of circular inclusion is considered when the spontaneous deformation is characterised by a deformation of the type given by $r^n \cos n\theta$. In

chapter IV we deal with the elliptic inclusion when the deformation is of the type $r^2 \cos 2\theta$. This is to indicate that this method may be used for spontaneous deformation of a general nature.

Chapter V deals with the problem when the matrix is undergoing spontaneous deformation and the inclusion is initially unstressed. This state may be created for example by taking a certain plane harmonic temperature distribution within the matrix with an insulation on the interface of the inclusion. This type of problems form a new class under such problems.

In chapter VI, to provide a coherent approach to two subsequent chapters(VII and VIII), necessary theory, first formulated by Tiffen ((21)) is given. In chapter VII the circular inclusion is considered in semi-infinite medium with its straight edge stress-free. Chapter VIII deals with the problem of circular inclusion in half plane when the leading edge is free from displacements.

In chapter IX the relevant theory of a point-force acting in the interior of an infinite elastic strip is given. This theory has been used in subsequent chapters. The theory is based on the work of Tiffen ((21, 22, 23, 24)). Chapter X and XI provide the solution of inclusion

problems in infinite elastic strip. In first case, the straight boundaries of the strip are traction-free whilst in the second case they are displacement free. It is found that the edge effect is confined to a small region around the inclusions and when the width of the strip is ten times the radius of circular inclusion the solutions differ slightly from those for the infinite case, the error being of the order of about six in hundred.

The work presented in the chapters III, IV and V is based on the following papers which have been published.

1. Circular Region under Plane Harmonic Temperature Distribution in an Insulated Infinite Elastic Medium. (Bulletin de l' Academie Polonaise de Sciences, Vol. XII, No. 7 1964).

2. An Elliptic Region under Plane Harmonic Temperature Distribution with Insulated Boundary (Bulletin de l' Academie Polonaise des Sciences, Vol. XII No. 12 1964).

3. An Infinite Elastic Medium under Plane Harmonic Temperature Distribution with a circular Insert. (Jour. Phys. Soc. of Japan, Vol. 19, No. 5, 1964).

LIST OF SYMBOLS

x, y	two-dimensional Cartesian coordinates
r, θ	two-dimensional polar coordinates
ξ, η	two-dimensional elliptic coordinates
u_x, u_y	displacement components in Cartesian coordinates
u_r, u_θ	displacement components in polar coordinates
e_{xx}, e_{xy}, e_{yy}	strain components in Cartesian coordinates
p_{xx}, p_{xy}, p_{yy}	stress components in Cartesian coordinates
$p_{rr}, p_{r\theta}, p_{\theta\theta}$	stress components in polar coordinates
$p_{\xi\xi}, p_{\xi\eta}, p_{\eta\eta}$	stress components in elliptic coordinates
ν	Poisson's ratio
λ, μ	Lame' constants
$\kappa = (3-\nu)/(1+\nu)$	for plane stress case
$\kappa = 3-4\nu$	for plane strain case
i	square root of -1
α	coefficient of linear expansion
T	temperature distribution
Subscript i	denotes that subscripted quantity pertains to inclusion
Subscript m	denotes that subscripted quantity pertains to matrix
Bar ($\bar{}$)	denotes the complex conjugate
Prime (')	denotes differentiation with respect to the argument

CHAPTER I

COMPLEX-VARIABLE APPROACH

This chapter summarises the complex-variable method of solving two-dimensional problems in infinitesimal theory of elasticity. This method of solution was first indicated by Kolosov ((1)) in 1908, and was developed in Russia extensively. Notable mention may be made of the classical book by Muskhelishvili ((2)). However the literature remained unknown for a long time (till the publication of I.S. Sokolnikoff's book ((4)) to the workers in west, and was independently discovered by Stevenson ((5)). The theory has also been discussed by Sokolonikoff ((4)) Green and Zerna ((7)) and Timoshenko and Goodier ((8)) et. al.

The solution of this class of problems depends upon two analytic functions of complex-variable All the formulas which will be needed in this thesis are included in what follows, for ready reference.

The attention shall be restricted to those plane strain problems for which the body forces are zero. In plane strain problems the axes canbe chosen in such a way that the displacement component in z direction is zero and other two displacement components are functions of x and y only. Thus the strain components e_{yz}, e_{zx}, e_{zz} are identically equal to zero and therefore by Hooke's law the stress component p_{yz}, p_{zx} are also zero. From $e_{zz} \equiv 0$, it may be noted that $p_{zz} = \nu(p_{xx} + p_{yy})$ where ν is the Poisson's ratio. All the remaining components of stress and strain are functions of x and y only.

The equilibrium equations in the absence of body forces are

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} = 0 \quad , \qquad \frac{\partial p_{yx}}{\partial x} + \frac{\partial p_{yy}}{\partial y} = 0 \, , \tag{1}$$

and it can be shown with the help of compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{2\partial^2 e_{xy}}{\partial x \partial y} \tag{2}$$

and (1), that $p_{xx} + p_{yy}$ is harmonic function. It may

be remarked, that in plane strain case, other compatibility relations are identically satisfied.

Noting that,

$$\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right) \quad , \quad \frac{\partial}{\partial y}=\left(\frac{i\partial}{\partial z}-\frac{i\partial}{\partial \bar{z}}\right) \quad ,$$

and $p_{xx}+p_{yy}$ satisfies Laplace's equation

$$\nabla^2(p_{xx}+p_{yy}) = 4\frac{\partial^2}{\partial z\partial \bar{z}}\left(p_{xx}+p_{yy}\right)=0. \qquad (3)$$

We at once obtain

$$p_{xx}+p_{yy} = 2\left(\Phi(z)+\overline{\Phi(z)}\right), \qquad (4)$$

where the factor 2 has been inserted for the sake of convenience and $\Phi(z)$ and $\overline{\Phi}(\bar{z})$ are complex conjugate functions.

Another relation involving p_{xx}, p_{xy} and p_{yy} may be obtained as follows :

Multiplying second of equations (1) by i subtracting it from the first equation and using (4), we obtain

$$\frac{\partial}{\partial \bar{z}}\left(p_{yy}-p_{xx}+2ip_{xy}\right)=\frac{\partial}{\partial z}\left(p_{xx}+p_{yy}\right)=2\Phi'(z), \qquad (5)$$

where dash denotes differentiation w.r.t. the argument inside the bracket.

Integration of (5) with respect to $\bar{z}$ gives immediately

$$p_{yy} - p_{xx} + 2i p_{xy} = 2\left[\bar{z}\,\Phi'(z) + \Psi(z)\right] \tag{6}$$

where $\Psi(z)$ is second function of z. It is thus obvious that the stresses, p_{xx}, p_{xy} and p_{xy} and also $p_{zz} = \nu(p_{xx} + p_{yy})$ may be represented in terms of two analytic functions $\phi(z)$ and $\psi(z)$ and of $\bar{z}$.

In order to obtain the corresponding expressions for displacements, Hooke's law connecting the strain to stress is used :

$$e_{xx} = \frac{1}{E}\left\{p_{xx} - \nu(p_{yy} + p_{zz})\right\}$$

$$e_{yy} = \frac{1}{E}\left\{p_{yy} - \nu(p_{zz} + p_{xx})\right\}$$

$$e_{xy} = \frac{p_{xy}}{2\mu} \tag{7}$$

where E is Young's modulus, and

$$\mu = \frac{E}{2(1+\nu)}.$$

Equation (7) combined with (4) and (6) would give

$$4\mu \frac{\partial}{\partial z}(u_x - iu_y) = -(p_{yy} - p_{xx} + 2ip_{xy}) = -2[\bar{z}\,\Phi'(z) + \Psi(z)]$$

and hence, after integrating with respect to z and taking the complex conjugate expression throughout,

$$2\mu(u_x + iu_y) = -z\bar{\Phi}(\bar{z}) - \int \bar{\Psi}(\bar{z})\,d\bar{z} + \chi(z) \quad , \qquad (8)$$

where $\chi(z)$ is at present is still undetermined. It may be shown with the help of (2) and (7) that

$$\chi(z) = \kappa \int \Phi(z)\,dz \qquad (9)$$

where $\kappa = 3 - 4\nu$.

Introducing two functions $\phi(z)$ and $\psi(z)$, defined by

$$\Phi(z) = \phi'(z) \qquad \Psi(z) = \psi'(z) \qquad (10)$$

the basic formulaes (4), (6) and (8) giving the complex representations of the stresses and displacements may be written in various equivalent forms

$$p_{xx} + p_{yy} = 2[\Phi(z) + \bar{\Phi}(\bar{z})] = 2[\phi'(z) + \bar{\phi}'(\bar{z})] \qquad (11a)$$

$$p_{yy} - p_{xx} + 2ip_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] = 2[\bar{z}\phi''(z) + \psi'(z)]. \qquad \text{(11b)}$$

$$2\mu(u_x + iu_y) = \kappa\phi(z) - z\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z}) \qquad \text{(11c)}$$

Finally, we obtain from first two of these equations by subtraction

$$p_{xx} - ip_{xy} = \Phi(z) + \bar{\Phi}(\bar{z}) - \bar{z}\Phi'(z) - \Psi(z).$$

If the axes x, y are rotated through an angle θ in the anticlockwise direction and the new axes denoted by axes x', y', the stresses $p_{x'x'}, p_{x'y'}, p_{y'y'}$ and $u_{x'}, u_{y'}$ referred to x', y' axes are related to p_{xx}, p_{xy}, p_{yy}; u_x, u_y referred to x, y, in the following manner

$$p_{x'x'} + p_{y'y'} = p_{xx} + p_{yy}$$

$$(p_{y'y'} - p_{x'x'} + 2ip_{x'y'}) = (p_{yy} - p_{xx} + 2ip_{xy})e^{2i\theta} \qquad \text{(12)}$$

and

$$(u_{x'} + iu_{y'}) = (u_x + iu_y)e^{-i\theta} \qquad \text{(13)}$$

So that if p_{nn} and p_{nt} are normal and tangential components of stress acting on a boundary at a point where outward normal makes an angle θ with the x-axis, then

$$2(p_{nn}-ip_{nt}) = p_{xx}+p_{yy}-(p_{yy}-p_{xx}+2ip_{xy})e^{2i\theta} \tag{14}$$

Substituting the values of $p_{xx}+p_{yy}$ and $p_{yy}-p_{xx}+2ip_{xy}$ from (11a) and (11b) in (14) gives

$$p_{nn}-ip_{nt} = \Phi(z)+\overline{\Phi}(\bar{z})-[\bar{z}\Phi'(z)+\Psi(z)]e^{2i\theta} \tag{15}$$

If the stresses p_{nn}, p_{nt} are prescribed on the boundary L, then z will be a point on L.

Now, we shall briefly discuss below the consequences of the changes of origin and of the rotation of axes on the functions $\phi(z)$ and $\psi(z)$, corresponding to a given state of stress of a body.

First investigate the effect of translation of the origin to a new point (x_0, y_0). Let (x, y) and (x_1, y_1) be the coordinates of the same point in the old and new systems.

Let

$$z_0 = x_0 + iy_0 \quad , \; z = x + iy \quad , \; z_1 = x_1 + iy_1 \quad .$$

It is obvious that

$$z = z_1 + z_0 \tag{16}$$

Now we start with the formulas (11a) and (11b), denote by $\Phi_1(z_1)$ and $\Psi_1(z_1)$ the functions playing in the new system the same role as $\Phi(z)$ and $\Psi(z)$ in the old one. Since the stress components are invariant to translation, one has by (11a)

$$\mathrm{Re}\left\{\Phi(z)\right\} = \mathrm{Re}\left\{\Phi_1(z_1)\right\} = \mathrm{Re}\,\Phi_1(z-z_0)$$

whence

$$\Phi(z) = \Phi_1(z-z_0) \,. \tag{17}$$

It may be remarked that the addition of a purely imaginary constant on the right hand side would have no influence on the stress distribution.

The formula (11b) gives

$$\bar{z}\,\Phi'(z) + \Psi(z) = \bar{z}_1\,\Phi_1'(z_1) + \Psi_1(z_1)$$

$$= (\bar{z} - \bar{z}_0)\,\Phi_1'(z - z_0) + \Psi_1(z - z_0)$$

$$= \bar{z}\,\Phi_1'(z - z_0) + \Psi_1(z - z_0) - \bar{z}_0\,\Phi_1'(z - z_0)$$

hence, by (17)

$$\Psi(z) = \Psi_1(z - z_0) - \bar{z}_0\,\Phi_1'(z - z_0) \tag{18}$$

Integrating (17) and (18) with respect to z one obtains

$$\begin{aligned} \phi(z) &= \phi_1(z - z_0) \\ \psi(z) &= \psi_1(z - z_0) - \bar{z}_0\,\phi_1'(z - z_0) \end{aligned} \tag{19}$$

Arbitrary constants which do not affect the stress-distribution have been omitted.

Next, consider the effect of rotation of axes, keeping the origin fixed. If the new axis Ox_1 makes an angle α with the axis Ox, then the point (x, y) in the (x, y) coordinate system is related to the point (x_1, y_1) in the (x_1, y_1) coordinates by the relation.

$$x = x_1 \cos\alpha - y_1 \sin\alpha\,,$$

$$y = x_1 \sin\alpha + y_1 \cos\alpha\,;$$

Therefore

$$x+iy = (x_1+iy_1)\,e^{i\alpha}$$

or

$$z = z_1 e^{i\alpha} \quad , \quad z_1 = z\,e^{-i\alpha} \tag{20}$$

Owing to the invariance of $p_{xx}+p_{yy}$, one has, on the basis of (11a)

$$\mathrm{Re}\,\Phi(z) = \mathrm{Re}\,\Phi_1(z_1) = \mathrm{Re}\,\Phi_1(z\,e^{-i\alpha})$$

whence, omitting a purely imaginary constant term,

$$\Phi(z) = \Phi_1(z\,e^{-i\alpha}) \tag{21}$$

Now using formula analogous to second of (12)

$$\bar{z}_1\,\Phi_1'(z_1) + \Psi_1(z_1) = \left[\bar{z}\,\Phi'(z) + \Psi(z)\right] e^{2i\alpha}$$

Thus

$$\bar{z}\,\Phi'(z) + \Psi(z) = \left[\bar{z}\,e^{i\alpha}\Phi_1'(z\,e^{-i\alpha}) + \Psi_1(z\,e^{-i\alpha})\right] e^{-2i\alpha}.$$

Further, noting that by (21) that

$$\Phi'(z) = e^{-i\alpha}\,\Phi_1'(z\,e^{-i\alpha})$$

one gets

$$\Psi(z) = \Psi_1(z\,e^{-i\alpha})\,e^{-2i\alpha}. \tag{22}$$

Integrating (21) and (22) with respect to z and omitting unnecessary arbitrary constants which do not influence the stress distribution, one obtains

$$\phi(z) = \phi_1(z e^{-i\alpha}) e^{i\alpha},$$

$$\psi(z) = \psi_1(z e^{-i\alpha}) e^{-i\alpha}. \tag{23}$$

CHAPTER II

INCLUSION PROBLEM AND POINT-FORCE

As this thesis deals with a class of inclusion problems in elasticity theory, a brief description of the problem is given. The method is explained with the help of the well known circular inclusion problem. The inclusion problem states that :

A region (the inclusion) of an elastic material tends to undergo a spontaneous change, which in the absence of the surrounding material (the matrix), of the elastic material, would be a prescribed homogeneous deformation. Stresses develop because of the constraints. The problem is to find the elastic field. Precise meanings to the terms used in this thesis are given below :

'Inclusion' is the region, which is deforming and is of the same material, as that of surrounding material, the matrix. Now as the inclusion undergoes a spontaneous

change in shape and size, the elastic constraints of the matrix will generate locked-up accomodation stresses everywhere within the inclusion and matrix. The determination of the resultant stress field and the equilibrium configuration form the subject matter of the inclusion problem.

The term ''free inclusion'' is used here after for the free state configuration which the inclusion would attain in the absence of the matrix.

On physical grounds, for a uniform expansion or contraction of sphere or a circle the equilibrium boundary is a similarity situated sphere or a circle and the problem can be easily solved e.g. Mott and Nabarro ((9)). The result is also true for ellipsoid ((10)) and elliptic boundaries ((13)). But the generalization is not possible. For instance, when the inclusion and free-inclusion are similarly situated rectangles, the equilibrium boundary is not a similar rectangle, in fact it is not a rectangle at all (Bhargava and Kapoor ((14)). Thus the equilibrium interface is unknown of the problem. However, a very powerful and ingenious method to solve such problems was given by Eshelby ((10)). It uses the results due to a point-force in an infinite medium. We briefly go over

the arguments which invokes a sequence of following hypothetical operations, and which solves the problem.

First cut out the inclusion from the medium and allow it to achieve free state configuration. Now, as it is, inclusion can no longer be fitted without straining into the cavity from which it was taken out. Next impress upon it the surface tractions, that restore its original dimensions. At this stage there will be a stress-field present in the inclusion. We shall call this stress-field as ''the constrained stress-field'' for future reference. Insert the stressed inclusion into the cavity left behind and rejoin the material across the cut. At this stage no stresses appear in the matrix. Finally, a distribution of point forces equal and opposite to the impressed surface tractions, is introduced on the boundary. If the matrix were absent, these forces would obviously nullify the surface tractions and would generate an elastic deformation which will exactly take the inclusion to its free state configuration. However, owing to the elastic constraints of the matrix, an elastic field would be produced in the matrix and an additional field in the inclusion.

Thus, if the stress field in a system, due to a concentrated force at a point, is known, the cumulative effect due to the distribution of point-forces can be found by integrating along the boundary. The stress-field in the matrix, due to the deforming inclusion, will be same as due to the distribution of point forces. In the inclusion, however, the stress field is obtained by superposing the stress-field due to the layer of point-forces, on that originally present due to the impressed surface tractions.

The problem of a concentrated force acting at a point in an infinite elastic medium was first discussed by Lord Kelvin ((16)). A force (X, Y, Z) acting at a point (x, y, z), produces a displacement (u, v, w) at (x, y, z) given by the formulas

$$u = \frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)}\left[\frac{\lambda+3\mu}{\lambda+\mu}\,\frac{X}{d} + (x-x_1)\left\{\frac{X(x-x_1)+Y(y-y_1)+Z(z-z_1)}{d^3}\right\}\right]$$

where

$$d^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$$

with similar expression for v and w.

For two-dimensional problems the expression for displacement at a point (x, y) due to a concentrated force acting at a point (x_1, y_1) is

$$2\mu u = -\frac{\kappa X}{\pi(1+\kappa)} \log\{(x-x_1)^2+(y-y_1)^2\}^{1/2}$$
$$+\frac{1}{2\pi(\kappa+1)}\left[\frac{X\{(x-x_1)^2-(y-y_1)^2\}+2(x-x_1)(y-y_1)}{(x-x_1)^2+(y-y_1)^2}\right]$$

$$2\mu v = -\frac{\kappa Y}{\pi(1+\kappa)} \log\{(x-x_1)^2+(y-y_1)^2\}^{1/2}$$
$$+\frac{1}{2\pi(\kappa+1)}\left[\frac{Y\{(y-y_1)^2-(x-x_1)^2\}+2(x-x_1)(y-y_1)}{(x-x_1)^2+(y-y_1)^2}\right]$$

where $\kappa = 3-4\nu$, for plane strain and $\kappa = \frac{3-\nu}{1+\nu}$ for the plane stress case.

A complex-variable formulation for such problem can be easily found out. As already stated in the last chapter, the elastic field is completely known, if two functions $\phi(z)$ and $\psi(z)$ are known. In the case of a concentrated force $P = X+iY$ acting at point ξ of an unbounded elastic body the complex potential functions $\phi(z)$ and $\psi(z)$ are given in Green and Zerna ((7)) as

$$\phi'(z) = -\frac{P}{2\pi(1+\kappa)}\,\frac{1}{(z-\xi)}$$

$$\psi'(z) = \frac{\kappa \bar{P}}{2\pi(1+\kappa)} \frac{1}{(z-\zeta)} - \frac{\bar{\zeta} P}{2\pi(1+\kappa)} \frac{1}{(z-\zeta)^2} , \tag{24}$$

where $\bar{P}$ is conjugate of P. The success of complex variable approach hinges on these key results.

The cumulative effect of distribution of point-forces acting along a simple arc Γ of an infinite elastic medium may be obtained by integrating the effects of the concentrated forces given as a function of ζ on Γ. Thus for concentrated forces acting along Γ, the functions $\phi(z)$ and $\psi(z)$ are given by

$$\phi'(z) = -\frac{1}{2\pi(\kappa+1)} \int_\Gamma \frac{P ds}{z-\zeta} ,$$

$$\psi'(z) = \frac{\kappa}{2\pi(\kappa+1)} \int_\Gamma \frac{\bar{P} ds}{z-\zeta} - \frac{1}{2\pi(\kappa+1)} \int_\Gamma \frac{\bar{\zeta} P ds}{(z-\zeta)^2} , \tag{25}$$

where ds denotes the arc differential along Γ. It may be emphasized that ζ lies on Γ. To evaluate integrals in equation (25) as functions of z, we write $P ds$ and $\bar{P} ds$ as follows : Let $\zeta = \xi + i\eta$ and therefore,

$$d\zeta = \left(\frac{d\xi}{ds} + i\frac{d\eta}{ds}\right) ds , \quad d\bar{\zeta} = \left(\frac{d\xi}{ds} - i\frac{d\eta}{ds}\right) ds$$

so that

$$\frac{d\xi}{ds} = \frac{1}{2}\left(\frac{d\zeta}{ds} + \frac{d\bar{\zeta}}{ds}\right) \quad , \quad \frac{d\eta}{ds} = -\frac{i}{2}\left(\frac{d\zeta}{ds} - \frac{d\bar{\zeta}}{ds}\right) \tag{26}$$

Now $d\bar{\zeta}$ may be removed by writing the equation of Γ in the form $\bar{\zeta} = f(\zeta)$

At the point (ξ, η) of an inclusion boundary Γ, the outward normal to Γ has direction cosines $\frac{d\eta}{ds}, -\frac{d\xi}{ds}$. Hence, if Eshelby's hypothetical stress-field is $p^{o}_{xx}, p^{o}_{xy}, p^{o}_{yy}$ the point-force components per unit length are

$$X = p^{o}_{xx}\left(\frac{d\eta}{ds}\right) + p^{o}_{xy}\left(-\frac{d\xi}{ds}\right)$$
$$Y = p^{o}_{xy}\left(\frac{d\eta}{ds}\right) + p^{o}_{yy}\left(-\frac{d\xi}{ds}\right)$$

Now, making use of equation (26) one can arrive at the expressions

$$P\,ds = -\frac{i}{2}\left[(p^{o}_{xx} + p^{o}_{yy})\,d\zeta - (p^{o}_{xx} - p^{o}_{yy})\,d\bar{\zeta}\right] + p^{o}_{xy}\,d\bar{\zeta}$$
$$\bar{P}\,ds = -\frac{i}{2}\left[(p^{o}_{xx} - p^{o}_{yy})\,d\zeta - (p^{o}_{xx} + p_{yy})\,d\bar{\zeta}\right] + p^{o}_{xy}\,d\zeta \tag{27}$$

Hence the expressions for $P\,ds$ and $\bar{P}\,ds$ in (25) are known.

As an illustration let us take a circular inclusion of unit radius in an infinite elastic medium. This tends

to expand to a size of radius $1+\delta$, in the absence of matrix, (where δ is small so that the linear theory of elasticity is applicable.) This is what has been termed as 'free inclusion'. At this stage, we reduce the inclusion to the size of the hole by applying surface tractions. The displacement field is given by

$$u_x = -\delta x \quad , \quad u_y = -\delta y$$

and therefore the strains $e_{xx} = -\delta$, $e_{yy} = -\delta$, $e_{xy} = 0$ and hence by Hooke's law

$$p_{xx} = -2(\lambda+\mu)\delta \ , \ p_{yy} = -2(\lambda+\mu)\delta \ , \ p_{xy} = 0 .$$

We leave this inclusion in the hole and apply surface tractions (which would have taken the inclusion to its free state in the absence of the matrix). This in effect generates a layer of point forces and are obtained from (27), by substituting the values of p_{xx} , p_{xy} and p_{yy} as given by above relation with negative sign, whence

$$P\,ds = -2i(\lambda+\mu)\delta\, d\xi \ , \ \bar{P}\,ds = 2i(\lambda+\mu)\delta\, d\bar{\xi} .$$

Substituting these values of $P\,ds$ and $\bar{P}\,ds$ in (26), evaluating the integral and noting that Γ is a circle of

unit radius, we get, for a point z in the inclusion

$$\phi_i'(z) = \frac{2(\lambda+\mu)\delta}{K+1}, \quad \psi_i'(z) = 0,$$

and for a point z in the matrix

$$\phi_m'(z) = 0, \quad \psi_m'(z) = \frac{K-1}{K+1}(\lambda+\mu)\frac{2\delta}{z^2}$$

where the potential functions $\phi_i'(z)$ and $\psi_i'(z)$; and $\phi_m'(z)$ and $\psi_m'(z)$ refer to the inclusion and matrix respectively.

The stresses and displacements in the matrix can be directly found from the corresponding expressions for complex potential functions with the aid of relations (11). But in case of the inclusion, the complex potential functions $\phi_i'(z)$ and $\psi_i'(z)$ give only a part of the stress-field. The constrained stress-field given by

$$p_{xx} = 2(\lambda+\mu)\delta, \quad p_{yy} = 2(\lambda+\mu)\delta, \quad p_{xy} = 0,$$

must be superposed to it to obtain the net stresses in the inclusion. Continuity of tangential and normal stresses across the boundary Γ is a check on the fore-going analysis. Similarly the net displacement in the inclusion is found by superposing the displacements obtained by the use of $\phi_i'(z)$, $\psi_i'(z)$ in (11c) over initial displacement field.

CHAPTER III

CIRCULAR INCLUSION WITH PLANE HARMONIC TEMPERATURE DISTRIBUTION

Previous works of Mott and Nabarro ((9)), Eshelby ((10, 11)) Jaswon and Bhargava ((13)), Bhargava and Radhakrishna ((17, 18)) and Bhargava and Kapoor ((14)) and of others have mainly been confined to the case of homogeneous spontaneous deformation. This was characterised by taking the spontaneous deformation as given by

$$u_x = \delta_1 x + \gamma_1 y \quad , \quad u_y = \delta_2 y + \gamma_1 x$$

In this chapter we shall consider a more general problem, where such a displacement is given by

$$\begin{aligned} u_x &= \delta_1 r^n \cos n\theta + \gamma_1 r^n \sin n\theta \\ u_y &= \delta_2 r^n \sin n\theta + \gamma_1 r^n \cos n\theta \end{aligned} \tag{28}$$

where n is a positive integer.

A physical meaning to such a spontaneous deformation can be given as follows : Consider the following problem :

A prism, of circular cross-section of radius a and centre at the origin is embedded into an infinite medium, and is insulated at the common interfare. The prism is subjected to the temperature distribution of the form

$$T(r,\theta) = b_n r^n \cos n\theta \text{ or } T(r,\theta) = b_n r^n \sin n\theta \tag{29}$$

It is obvious that such a temperature distribution is satisfying two-dimensional Laplace equation (Boley and Wiener ((19)),

$$\nabla^2 T(r,\theta) = \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0. \tag{30}$$

Due to this temperature distribution, there would be a free expansion in the inclusion, but for the constraints of matrix. Hence the stresses would develop both in inclusion and in matrix. The problem is to find this elastic field.

It may be remarked that such a temperature distribution

can cater for the following still more general problem. Consider an arbitrary temperature distribution. Suppose that temperature distribution is expressed in Fourier series as

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \tag{31}$$

This obviously satisfies Laplace equation. If the results are derived for (29), the results for the problem (31) can be derived by superposition.

This problem may be solved directly by the following hypothetical considerations :

The inclusion is cut out from the matrix and is allowed to undergo the temperature distribution. This would change its dimensions. Surface tractions are applied to bring the inclusion back to its initial shape and size. We put it back into the hole of the matrix, and the operations similar to those indicated in chapter II page 14 are applied.

The stress-strain relations of thermoelasticity were first given by Duhamel ((20)). The stresses in the absence of matrix can easily be seen to be

$$p_{xx} = p_{yy} = 2\alpha(\lambda+\mu)T$$

$$= KT \quad , \quad p_{xy} = 0. \tag{32}$$

where $K = 2\alpha(\lambda+\mu)$ and α is the coefficient of linear expansion of the material under consideration. Substitution of these values of stresses p_{xx} , p_{xy} , p_{yy} in equations (27) the following equation is obtained

$$P\,ds = -iKT\,d\xi \quad , \quad \bar{P}\,ds = iKT\,d\bar{\xi}. \tag{33}$$

It may be noted that at any point $\xi = re^{i\theta}$ where r is the radius vector and θ is the vectorial angle.

We shall first consider the case when the temperature distribution is

$$T(r,\theta) = b_n r^n \cos n\theta = \frac{b_n}{2}\left\{ \xi^n + \bar{\xi}^n \right\} \tag{34}$$

If the point is taken on the boundary, ξ will be written as σ thus the equation of circular boundary Γ is $\sigma\bar{\sigma} = a^2$, hence $\bar{\sigma} = a^2/\sigma$ on the boundary where $\bar{\sigma}$ is the complex conjugate of σ . Putting the value of T

from (34) in (33) and then substituting the value of $p\,ds$ in (26), and integrating the resultant contour integrals around the circle Γ, two values of each $\phi'(z)$ and $\psi'(z)$ are obtained depending upon whether z lies inside or outside the circle Γ. After some simple calculation, the following values are obtained

$$\phi_i'(z) = \frac{K b_n z^n}{2(1+K)} ,$$

$$\psi_i'(z) = -\frac{K b_n a^2 (K+n-1)}{2(K+1)} z^{n-2} ; \tag{35}$$

$$\phi_m'(z) = -\frac{K b_n a^{2n}}{2(K+1)} \frac{1}{z^n} ,$$

$$\psi_m'(z) = \frac{K b_n a^{2n+2} (K-n+1)}{2(K+1)} \frac{1}{z^{n+2}} . \tag{36}$$

For evaluation of stresses, the values of complex potential functions are substituted from (35) - (36). It may, however, be noted that the inclusion had an initial stress-field termed previously as 'constrained stress-field.' Hence for finding out the actual stresses the constrained field has to be superposed upon that obtained with the help of functions $\phi_i(z)$ and $\psi_i(z)$. Following the procedure outlined above, we shall get after

some simplification that stresses in the inclusion are

$$p_{rr} = \frac{K b_n r^{n-2} \cos n\theta}{2(1+K)} \left\{ -nr^2 - 2Kr^2 + a^2(K+n-1) \right\}$$

$$p_{\theta\theta} = \frac{K b_n r^{n-2} \cos n\theta}{2(1+K)} \left\{ nr^2 - 2Kr^2 - a^2(K+n-1) \right\} \tag{37}$$

and

$$p_{r\theta} = \frac{K b_n r^{n-2} \sin n\theta}{2(1+K)} \left\{ a^2(1-K) - n(a^2-r^2) \right\}. \tag{38}$$

Here we use small letters p_{rr} , $p_{r\theta}$, $p_{\theta\theta}$ to denote the stresses in the inclusion. The capital letter would be used for corresponding quantities in the matrix. Thus P_{rr} , $P_{r\theta}$, $P_{\theta\theta}$ will refer to the stresses in the matrix. For denoting the boundary values of these quantities we shall use the superscript b . The boundary stresses are

$$p^b_{rr} = -\frac{K b_n a^n \cos n\theta}{2}$$

$$p^b_{\theta\theta} = \frac{K b_n a^n \cos n\theta}{2} \left(\frac{1-3K}{1+K}\right) \tag{39}$$

$$p^b_{r\theta} = \frac{K b_n a^n \sin n\theta}{2} \left(\frac{1-K}{1+K}\right)$$

The complex potential function (36) with (12), give the stress-field in the matrix, to be

$$P_{rr} = -\frac{K b_n a^{2n} \cos n\theta}{2(1+K)\, r^{n+2}} \left(2r^2 + K a^2 - a^2 \right),$$

$$P_{\theta\theta} = -\frac{K b_n a^{2n} \cos n\theta}{2(1+K)\, r^{n+2}} \left(2r^2 - K a^2 + a^2 \right), \tag{40}$$

and

$$P_{r\theta} = \frac{K b_n a^{2n+2} \sin n\theta}{2\, r^{n+2}} \left(\frac{1-K}{1+K} \right). \tag{41}$$

Boundary values of these stresses are

$$P^b_{rr} = -\frac{K b_n a^n \cos n\theta}{2},$$

$$P^b_{\theta\theta} = \frac{-K b_n a^n \cos n\theta}{2} \left(\frac{3-K}{1+K} \right), \tag{42}$$

$$P^b_{r\theta} = \frac{K b_n a^n \sin n\theta}{2} \left(\frac{1-K}{1+K} \right).$$

From the expressions (39) and (42) we observe that the normal and tangential components of stress are continuous on the equilibrium interface, which it should be. The hoop-stresses have a jump-discontinuity over the boundary.

The jump is

$$P^{b}_{\theta\theta} - p^{b}_{\theta\theta} = K b_n a^n \cos n\theta \tag{43}$$

The displacement field is worked out with the help of equations (11c) and (35) - (36). Thus the net displacement field of the inclusion (made up for elastic and non-elastic displacements) is obtained as

$$2\mu(u_x + i u_y) = \frac{K b_n \kappa z^{n+1}}{2(K+1)(n+1)} - \frac{K b_n z \bar{z}^n}{2(K+1)} + \frac{K b_n (\kappa+n-1) a^2 \bar{z}^{n-1}}{2(K+1)(n-1)} \tag{44}$$

The displacement components may be transformed from Cartesian to polar coordinates with the help of relations (13). The boundary-value of $u_r + i u_\theta$ for inclusion is given by

$$2\mu(u^b_r + i u^b_\theta) = \frac{K b_n \kappa a^{n+1}}{2(K+1)} \left[\frac{e^{ni\theta}}{n+1} + \frac{e^{-ni\theta}}{n-1} \right] \tag{45}$$

where we have again used small letters u_r , u_θ for the displacement in the inclusion and similarly for matrix we shall use capital letters.

The displacement in the matrix, is given by the relation (11c) and equation (36), as

$$2\mu(U_x+iU_y) = \frac{K b_n \kappa a^{2n}}{2(K+1)(n-1)\, z^{n-1}} + \frac{K b_n z a^{2n}}{2(K+1)\,\bar{z}^{n}}$$

$$-\frac{K b_n a^{2n+2}(\kappa-n-1)}{2(K+1)(n+1)\,\bar{z}^{n+1}} \,. \tag{46}$$

The components of displacements in polar coordinates at the boundary are

$$2\mu(U_r^b+iU_\theta^b) = \frac{K b_n \kappa a^{n+1}}{2(K+1)}\left[\frac{e^{ni\theta}}{n+1} + \frac{e^{-ni\theta}}{n-1}\right] \tag{47}$$

From (46) and (47), the continuity of displacement field over the interface is established.

The strain energy density of a two-dimensional elastic system per unit height in plane strain case is given by

$$W = \tfrac{1}{2} p_{ij} e_{ij} = \tfrac{1}{2}\left(p_{xx}e_{xx} + p_{yy}e_{yy} + 2p_{xy}e_{xy}\right)$$

which may be put in terms of stresses only by using Hooke's law (7). The strain energies in the inclusion and the matrix are

$$W_i = \frac{1}{4}\left[(p_{rr}+p_{\theta\theta})^2\frac{(1+\nu)(1-2\nu)}{E} + 2\alpha T(p_{rr}+p_{\theta\theta}) + (p_{\theta\theta}-p_{rr})^2\frac{1+\nu}{E} + \frac{2}{\mu}p_{r\theta}^2\right]$$

$$W_m = \frac{1}{4}\left[(p_{rr}+p_{\theta\theta})^2\frac{(1+\nu)(1-2\nu)}{E} + (p_{\theta\theta}-p_{rr})^2\frac{1+\nu}{E} + \frac{2}{\mu}p_{r\theta}^2\right] \tag{48}$$

By integrating across the area, the expressions for strain energy in the inclusion and the matrix are

$$W_i = \frac{\pi (K\, b_n a^n)^2}{16(\lambda+2\mu)^2 \times 2\mu n(n^2-1)} \Big[(\lambda+3\mu)\{\lambda n(n+1)+\mu(n+5)\}$$

$$+ (\lambda+\mu)^2\{-2n^4+n^2-n-2\} + 2(\lambda+\mu)(\lambda+3\mu)(n^3+1) \Big] \tag{49}$$

$$W_m = \frac{\pi (K\, b_n a^n)^2}{16(\lambda+2\mu)^2(n^2-1)} \Big[(\lambda+\mu)(n+1) + \mu(n-1) \Big]$$

Thus

$$\frac{W_i}{W_m} = \frac{\Big[(\lambda+3\mu)\{\lambda n(n+1)+\mu(n+5)\} + (\lambda+\mu)^2\{-2n^4+n^2-n-2\} + 2(\lambda+\mu)(\lambda+3\mu)(n^3+1) \Big]}{2n\mu\{(\lambda+\mu)(n+1)+\mu(n-1)\}} \tag{50}$$

for given value of n, the values of W_i/W_m can be easily computed from the above formulae.

A similar procedure is adopted to solve the problem, when the temperature distribution is of the type,

$$T(r,\theta) = b_n r^n \sin n\theta , \tag{51}$$

The following results are derived

$$\phi_i'(z) = -\frac{i K b_n z^n}{2(K+1)} ,$$

$$\psi_i'(z) = \frac{i K a^2 (K+n-1) b_n z^{n-2}}{2(K+1)} . \tag{52}$$

$$\phi'_m(z) = -\frac{iKb_n a^{2n}}{2(K+1)} \frac{1}{z^n}$$

$$\psi'_m(z) = \frac{iKb_n (K-n-1)\, a^{2n+2}}{2(K+1)\, z^{n+2}} \tag{53}$$

The stress-field in the inclusion is given by

$$p_{rr} = \frac{Kb_n r^{n-2} \sin n\theta}{2(K+1)} \left\{ -nr^2 - 2Kr^2 + a^2(K+n-1) \right\},$$

$$p_{\theta\theta} = \frac{Kb_n r^{n-2} \sin n\theta}{2(K+1)} \left\{ -2Kr^2 + nr^2 - a^2(n+K-1) \right\}, \tag{54}$$

$$p_{r\theta} = \frac{Kb_n r^{n-2} \cos n\theta}{2(K+1)} \left\{ nr^2 - a^2(K+n-1) \right\};$$

and in the matrix, it is given by

$$p_{rr} = \frac{-Kb_n a^{2n} \sin n\theta \left\{ 2r^2 + a^2(1+K) \right\}}{2(K+1)\, r^{n+2}},$$

$$p_{\theta\theta} = -\frac{Kb_n a^{2n} \sin n\theta}{2(1+K)\, r^{n+2}} \left\{ 2r^2 + a^2(1-K) \right\} \tag{55}$$

$$p_{r\theta} = \frac{Kb_n a^{2n+2}}{2} \frac{\cos n\theta}{r^{n+2}} \left(\frac{1-K}{1+K} \right).$$

The common displacement field over the common interface is

$$2\mu(u_r^b + iu_\theta^b) = -\frac{iKb_n K a^{n+1}}{2(K+1)} \left[\frac{e^{ni\theta}}{n+1} - \frac{e^{-ni\theta}}{n-1} \right]. \tag{56}$$

CHAPTER IV

ELLIPTIC INCLUSION WITH PLANE HARMONIC TEMPERATURE-DISTRIBUTION

This chapter deals with the problem of an elliptic region within a homogeneous elastic medium. The region is subjected to a particular type of temperature-distribution with common interface. The temperature distribution is of the form

$$T(r,\theta) = b_2 r^2 \cos 2\theta \tag{57}$$

where b_2 is constant and r and θ are polar coordinates. This type of temperature-distribution obviously satisfies steady state heat conduction equation in polar form given in equation (30).

It may be remarked that the previous work ((10)) - ((18)) on such problems related mainly to the cases when the temperature was constant throughout the inclusion.

It was characterised by the fact that the spontaneous displacement in the inclusion was of the form $u_x = \delta_1 x + \delta_3 y$, $u_y = \delta_2 y + \delta_3 x$. Although it would be more desirable to consider the case when $T = b_n r^n \cos n\theta$ or for $T(r,\theta) = b_n r^n \sin n\theta$, but because of mathematical complexities involved, the temperature distribution has been taken of the form given by the equation (67). Following a similar procedure it is possible to solve the problem for the general case $T = b_n r^n \cos n\theta$, numerically.

The solution of the problem may be obtained by the method explained in the beginning of the previous chapter.

The boundary conditions of the problem are that the normal and tangential components of the stress on the boundary shall be continuous. The stresses at infinity tend to zero at least as $O(1/r^2)$. The displacement field, made up of elastic and non-elastic contributions, should everywhere be continuous.

The formulae for $\Phi'(z)$ and $\Psi'(z)$ for a concentrated force P acting at a point z are given by equations (24). The cumulative effect of point forces is found out by integrating the effect of point-force at z over the

boundary Γ . Here Γ is the elliptic boundary $x^2/a^2 + y^2/b^2 = 1$. Now for convenience in mathematical formulation the equation Γ is written in the form $\bar{z} = f(z)$. For the elliptic case this equation is

$$\bar{z} = \frac{a^2+b^2}{c^2} z - \frac{2ab}{c^2}\sqrt{z^2-c^2} \quad , (c^2 = a^2-b^2) \tag{58}$$

With the help of thermoelastic stress-strain relationship the stresses in the absence of matrix are given by

$$p_{xx} = p_{yy} = KT \ , \quad p_{xy} = 0 \ , \tag{59}$$

where $K = 2\alpha(\lambda+\mu)$; α being coefficient of linear expansion and λ and μ are well known Lamé constants. Substitution of these values of p_{xx} , p_{xy} and p_{yy} in equations (27), would furnish the expressions for forces acting on arc ds . Thus

$$P_{ds} = -iKT\,d\xi \qquad \bar{P}_{ds} = iKT\,d\bar{\xi} \tag{60}$$

It may be noted that

$$T = \frac{b_2(z^2+\bar{z}^2)}{2}$$

where $\bar{z}$ is complex conjugate of z, defined by equation (68). These values of $P ds$ and $\bar{P} ds$ are substituted in equations (26). It may be noted that Γ is the elliptic boundary. Two values of each $\phi'(z)$ and $\psi'(z)$ are obtained, depending upon whether the point z is interier to the ellipse i.e. a point in the inclusion or exterior to the ellipse i.e. a point in the matrix.

After some calculation, it is seen that

$$\phi_i'(z) = \frac{K b_2}{2(K+1)} \left[\frac{2(a^2+b^2)}{(a+b)^2} z^2 + \frac{2ab(a-b)}{(a+b)} \right]$$

$$\psi_i'(z) = \frac{K b_2}{2(K+1)} \left[\frac{2(K-3)(a^2+b^2)(a-b)}{(a+b)^3} z^2 - \right.$$

$$\left. - \frac{4ab}{c^4}\left(a^4+b^4+4a^2b^2-3a^3b-3ab^3\right) - \frac{4a^2b^2K}{(a+b)^2} \right] \qquad (61)$$

for the inclusion; and

$$\phi_m'(z) = \frac{K b_2}{2(K+1)} \left[\frac{2ab(a^2+b^2)}{c^4} \left\{ 2z\sqrt{z^2-c^2} - (2z^2-c^2) \right\} \right]$$

$$\psi_m'(z) = \frac{K b_2}{2(K+1)} \left[\frac{1}{c^6} \left\{ 8ab(a^2+b^2)^2(3-K)\, z^2 + 4ab(a^2+b^2)^2 \left(2z\sqrt{z^2-c^2} + z^3/\sqrt{z^2-c^2}\right)(K-2) \right\} \right.$$

$$\left. + \frac{1}{c^4} \left\{ 8a^3b^3(1-K)\left(z/\sqrt{z^2-c^2} - 1\right) - 4ab(a^2+b^2)^2 \right\} \right] \tag{62}$$

for the matrix.

For the evaluation of stresses values of $\phi'(z)$ and $\psi'(z)$ in (61) and (62) are substituted in (11a) and (11b). It may, however be noted here that the inclusion has got 'constrained stress-field' given by

$$p_{xx} = -KT \quad , \quad p_{yy} = -KT \quad , \quad p_{xy} = 0 \tag{63}$$

Hence for finding out actual stresses, the 'constrained stress-field' has to be superposed upon that obtained by complex-potential (61).

At this stage it is more convenient to work with confocal elliptic coordinates ξ, η defined by the transformation

$$z = c\cosh(\xi + i\eta) \tag{64}$$

The stress and displacements components $p_{\xi\xi}, p_{\xi\eta}, p_{\eta\eta}$; u_ξ, u_η referred to ξ, η, the axes oriented at an angle θ to x-axis are related to Cartesian component, p_{xx}, p_{xy}, p_{yy}; u_x, u_y by the following relations

$$p_{\xi\xi} + p_{\eta\eta} = p_{xx} + p_{yy}$$

$$p_{\eta\eta} - p_{\xi\xi} + 2ip_{\xi\eta} = (p_{yy} - p_{xx} + 2ip_{xy})\,e^{2i\theta} \tag{65}$$

$$u_\xi + iu_\eta = (u_x + iu_y)\,e^{-i\theta}$$

In the present case θ denotes the angle between the x-axis and the normal at (ξ, η) (in the direction of increasing ξ). It may be seen that $e^{2i\theta} = \sinh(\xi+i\eta)/\sinh(\xi-i\eta)$. After some simplification, we get the actual stress components as :

$$p_{\xi\xi} = \frac{K b_2 c^2}{2(K+1)} \Big[-(1+K)\cosh 2\xi \cos 2\eta\,(\cosh 2\xi - \cos 2\eta)$$

$$-K(\cosh 2\xi - \cos 2\eta) + \frac{2(a^2+b^2)}{(a+b)^2}\left\{\cosh 2\xi + \cos 2\eta - 2\cos^2 2\eta \cosh 2\xi\right\}$$

$$+\frac{(K-3)(a^2+b^2)(a-b)}{(a+b)^3}\left\{\sinh^2 2\xi \cos^2 2\eta - \cosh^2 2\xi \sin^2 2\eta\right\}$$

$$+\frac{4ab}{c^6}\left\{a^4+b^4+4a^2b^2-3a^3b-3ab^3+Kab(a-b)^2\right\}\times$$

$$\times\left\{\cosh 2\xi \cos 2\eta - 1\right\} + \cosh 2\xi - \cos 2\eta\Big] \Big/ (\cosh 2\xi - \cos 2\eta).$$

$$p_{\eta\eta} = \frac{K b_2 c^2}{2(K+1)} \Big[-(1+K)\cosh 2\xi \cos 2\eta\,(\cosh 2\xi - \cos 2\eta)$$

$$-K(\cosh 2\xi - \cos 2\eta) + \frac{2(a^2+b^2)}{(a+b)^2}\left\{2\cosh^2 2\xi \cos 2\eta - \cosh 2\xi - \cos 2\eta\right\}$$

$$+\frac{(K-3)(a^2+b^2)(a-b)}{(a+b)^3}\left\{\sinh^2 2\xi \cos^2 2\eta - \cosh^2 2\xi \sin^2 2\eta\right\}$$

$$-\frac{4ab}{c^6}\left\{a^4+b^4+4a^2b^2-3a^3b-3ab^3+Kab(a-b)^2\right\}\left\{\cosh 2\xi \cos 2\eta - 1\right\}$$

$$+\cosh 2\xi - \cos 2\eta\Big] \Big/ (\cosh 2\xi - \cos 2\eta). \tag{66}$$

and

$$p_{\xi\eta} = \frac{Kb_2c^2}{2(K+1)} \left[\frac{2(a^2+b^2)}{(a+b)^2} \left\{ \cosh 2\xi \sinh 2\xi \sin 2\eta + \sinh 2\xi \cos 2\eta \sin 2\eta \right\} \right.$$

$$+ \frac{2(K-3)(a^2+b^2)(a-b)}{(a+b)^3} \cosh 2\xi \sinh 2\xi \cos 2\eta \sin 2\eta$$

$$- \frac{4ab}{c^6} \left\{ a^4 + b^4 + 4a^2b^2 - 3a^3b - 3ab^3 + Kab(a-b)^2 \right\} \times$$

$$\times \left\{ \sinh 2\xi \sin 2\eta \right\} \Big/ (\cosh 2\xi - \cos 2\eta)$$

(67)

Along the boundary ξ has the value ξ_0 and,

$$\cosh 2\xi_0 = \frac{a^2+b^2}{c^2} \quad , \quad \sinh 2\xi_0 = \frac{2ab}{c^2}$$

and the boundary stresses recognized by superscript b, after some simplification are

$$p^b_{\xi\xi} = \frac{Kb_2c^2}{2(K+1)} \left[\cos^2 2\eta \left\{ \cosh^5 2\xi_0 - 4\cosh^4 2\xi_0 \sinh 2\xi_0 \right. \right.$$

$$+ 5\cosh^3 2\xi_0 \sinh^2 2\xi_0 - 2\cosh^3 2\xi_0 - 2\cosh^2 2\xi_0 \sinh^3 2\xi_0 +$$

$$+2\cosh^2 2\xi_0 \sinh 2\xi_0 + \cosh 2\xi_0 + K\left(-\cosh^5 2\xi_0 + 2\cosh^4 2\xi_0 \sinh 2\xi_0\right.$$

$$\left.\left. - \cosh^3 2\xi_0 \sinh^2 2\xi_0 + \cosh 2\xi_0\right)\right\} + \sin^2 2\eta \left\{ - 3\cosh^3 2\xi_0 \sinh^2 2\xi_0\right.$$

$$+ 6\cosh^2 2\xi_0 \sinh^3 2\xi_0 - 3\cosh 2\xi_0 \sinh^4 2\xi_0 + K\left(\cosh^3 2\xi_0 \sinh^2 2\xi_0\right.$$

$$\left.\left. - 2\cosh^2 2\xi_0 \sinh^3 2\xi_0 + \cosh 2\xi_0 \sinh^4 2\xi_0\right)\right\} + \cos 2\eta \left\{ 2\cosh^4 2\xi_0 - \right.$$

$$- 5\cosh^2 2\xi_0 \sinh^2 2\xi_0 - \cosh^2 2\xi_0 + 3\cosh 2\xi_0 \sinh^3 2\xi_0 - 1$$

$$\left. + K\left(\cosh^2 2\xi_0 \sinh^2 2\xi_0 - \cosh 2\xi_0 \sinh^3 2\xi_0 - \cosh^2 2\xi_0 + 1\right)\right\}$$

$$+ K\left\{\cosh^3 2\xi_0 - 2\cosh^2 2\xi_0 \sinh 2\xi_0 - \cosh 2\xi_0 + \sinh^3 2\xi_0\right\} - \cosh^3 2\xi_0 \qquad (68)$$

$$\left. + 2\cosh^2 2\xi_0 \sinh 2\xi_0 + \cosh 2\xi_0 - \sinh^3 2\xi_0 \right] \Big/ \left(\cosh 2\xi_0 - \cos 2\eta\right)$$

$$p^b_{\eta\eta} = \frac{K b_2 c^2}{2(K+1)} \left[\cos^2 2\eta \left\{ - \cosh^5 2\xi_0 + 4\cosh^4 2\xi_0 \sinh 2\xi_0 - 5\cosh^3 2\xi_0 \sinh^2 2\xi_0 \right.\right.$$

$$- 2\cosh^3 2\xi_0 + 2\cosh^2 2\xi_0 \sinh^3 2\xi_0 + 2\cosh^2 2\xi_0 \sinh 2\xi_0 + \cosh 2\xi_0 +$$

$$+K\left(\cosh^5 2\xi_0 - 2\cosh^4 2\xi_0 \sinh 2\xi_0 + \cosh^3 2\xi_0 \sinh^2 2\xi_0 + \cosh 2\xi_0\right)\Big\}$$

$$+\sin^2 2\eta\Big\{3\cosh^3 2\xi_0 \sinh^2 2\xi_0 - 6\cosh^2 2\xi_0 \sinh^3 2\xi_0 + 3\cosh 2\xi_0 \sinh^4 2\xi_0 +$$

$$+K\left(-\cosh^3 2\xi_0 \sinh^2 2\xi_0 + 2\cosh^2 2\xi_0 \sinh^3 2\xi_0 - \cosh 2\xi_0 \sinh^4 2\xi_0\right)\Big\}$$

$$+\cos 2\eta\Big\{2\cosh^4 2\xi_0 - 4\cosh^3 2\xi_0 \sinh 2\xi_0 + 5\cosh^2 2\xi_0 \sinh^2 2\xi_0 - \cosh^2 2\xi_0 - 3\cosh 2\xi_0 \sinh^3 2\xi_0 - 1$$

$$+K\left(-\cosh^2 2\xi_0 \sinh^2 2\xi_0 - \cosh^2 2\xi_0 + \cosh 2\xi_0 \sinh^3 2\xi_0 + 1\right)\Big\} + K\Big\{-\cosh^3 2\xi_0 + 2\cosh^2 2\xi_0 \sinh 2\xi_0$$

$$-\cosh 2\xi_0 - \sinh^3 2\xi_0\Big\} + \cosh^3 2\xi_0 - 2\cosh^2 2\xi_0 \sinh 2\xi_0 + \cosh 2\xi_0 + \sinh^3 2\xi_0\Big] \Big/ \left(\cosh 2\xi_0 - \cos 2\eta\right)$$

$$p^b_{\xi\eta} = \frac{Kb_2c^2}{2(K+1)}\Big[\cos 2\eta \sin 2\eta\Big\{-8\cosh^4 2\xi_0 \sinh 2\xi_0 + 20\cosh^3 2\xi_0 \sinh^2 2\xi_0 - 16\cosh^2 2\xi_0 \sinh^3 2\xi_0$$

$$+4\cosh 2\xi_0 \sinh^4 2\xi_0 + K\left(4\cosh^4 2\xi_0 \sinh 2\xi_0 - 8\cosh^3 2\xi_0 \sinh^2 2\xi_0 + 4\cosh^2 2\xi_0 \sinh^3 2\xi_0\right)\Big\}$$

$$+\sin 2\eta\Big\{4\cosh^3 2\xi_0 \sinh 2\xi_0 - 8\cosh^2 2\xi_0 \sinh^2 2\xi_0 + 6\cosh 2\xi_0 \sinh^3 2\xi_0$$

$$-2\sinh^4 2\xi_0 + K\left(-2\cosh 2\xi_0 \sinh^3 2\xi_0 + 2\cosh^4 2\xi_0\right)\Big\}\Big] \Big/ \left(\cosh 2\xi_0 - \cos 2\eta\right)$$

As regards the matrix, the use of corresponding complex potentials $\phi'_m(z)$ and $\psi'_m(z)$ gives the expressions of stress components.

$$P_{\xi\xi} = \frac{Kb_2c^2}{2(K+1)}\left[\frac{2ab(a^2+b^2)}{c^4}\left\{4\cosh 2\xi\cos^2 2\eta - 3\sinh 2\xi\cos^2 2\eta - \cosh 2\xi\cos 2\eta\sin 2\eta\right.\right.$$

$$\left. + \sinh 2\xi + \sin 2\eta - 2\cosh 2\xi - 2\cos 2\eta\right\} - \frac{4ab(a^2+b^2)^2}{c^6}(3-K)\left\{\sinh^2 2\xi\cos^2 2\eta\right.$$

$$\left. - \cosh^2 2\xi\sin^2 2\eta\right\} - \frac{2ab(a^2+b^2)^2}{c^6}(K-2)\left\{3\cosh 2\xi\sinh 2\xi(\cos^2 2\eta - \sin^2 2\eta)\right\}$$

$$\left. - \frac{8a^3b^3}{c^6}(1-K)\left\{\cos 2\eta(\sinh 2\xi - \cosh 2\xi) + 1\right\} - \frac{4ab(a^2+b^2)^2}{c^6}(1 - \cosh 2\xi\cos 2\eta)\right] \Big/ (\cosh 2\xi - \cos 2\eta)$$

$$P_{\eta\eta} = \frac{Kb_2c^2}{2(K+1)}\left[\frac{2ab(a^2+b^2)}{c^4}\left\{-\cos^2 2\eta\sinh 2\xi + \cosh 2\xi\cos 2\eta\sin 2\eta + 4\cosh 2\xi\cos 2\eta \times\right.\right.$$

$$\left. \times(\sinh 2\xi - \cosh 2\xi) - \sinh 2\xi - \sin 2\eta + 2\cosh 2\xi + 2\cos 2\eta\right\} + \frac{4ab(a^2+b^2)^2}{c^6}(3-K)\times$$

$$\times\left\{\sinh^2 2\xi\cos^2 2\eta - \cosh^2 2\xi\sin^2 2\eta\right\} + \frac{2ab(a^2+b^2)^2}{c^6}(K-2)\left\{3\cosh 2\xi\sinh 2\xi(\cos^2 2\eta - \sin^2 2\eta)\right\}$$

$$+ \frac{8a^3b^3}{c^6}(1-K)\left\{\cos 2\eta(\sinh 2\xi - \cosh 2\xi) + 1\right\}$$

$$\left. + \frac{4ab(a^2+b^2)^2}{c^6}\left\{1 - \cosh 2\xi\cos 2\eta\right\}\right] \Big/ (\cosh 2\xi - \cos 2\eta) \qquad (60)$$

and

$$P_{\xi\eta} = \frac{Kb_2c^2}{2(K+1)}\Big[\frac{2ab(a^2+b^2)}{c^4}\{(\sinh 2\xi + \sin 2\eta - 2\cosh 2\xi - 2\cos 2\eta)\sinh 2\xi \sin 2\eta +$$

$$+ \cosh 2\xi \sin 2\eta(\cosh 2\xi + \cos 2\eta)\} + \frac{8ab(a^2+b^2)^2}{c^6}(3-K)\cosh 2\xi \sinh 2\xi \cos 2\eta \sin 2\eta$$

$$+ \frac{2ab(a^2+b^2)^2}{c^6}(K-2)\{3\cosh^2 2\xi \sinh^2 2\xi \cos 2\eta \sin 2\eta - \cosh 2\xi \sin 2\eta\} + \frac{8a^3b^3(1-K)}{c^6} \times$$

$$\times \sin 2\eta(\cosh 2\xi - \sinh 2\xi) - \frac{4ab(a^2+b^2)^2}{c^6}\sinh 2\xi \sin 2\eta\Big] \Big/ (\cosh 2\xi - \cos 2\eta) \tag{70}$$

The boundary value of these may be obtained by putting $\xi = \xi_0$ and using relations (70); it may be seen that

$$P^b_{\xi\xi} = p^b_{\xi\xi} \quad ; \quad P^b_{\xi\eta} = p^b_{\xi\eta}$$

The hoop stress $P^b_{\eta\eta}$ is given by

$$P^b_{\eta\eta} = \frac{Kb_2c^2}{2(K+1)}\Big[\cos^2 2\eta\{4\cosh^4 2\xi_0 \sinh 2\xi_0 - 6\cosh^3 2\xi_0 \sinh^2 2\xi_0 + 2\cosh^2 2\xi_0 \sinh^3 2\xi_0$$

$$+ 2\cosh^2 2\xi_0 \sinh 2\xi_0 - 2\cosh 2\xi_0 \sinh^2 2\xi_0 + K(-2\cosh^3 2\xi_0 \sinh^2 2\xi_0$$

$$+ 2\cosh^2 2\xi_0 \sinh^3 2\xi_0)\} + \sin^2 2\eta\{4\cosh^3 2\xi_0 \sinh^2 2\xi_0 - 6\cosh^2 2\xi_0 \sinh^3 2\xi_0 +$$

$$+ 2\cosh 2\xi_0 \sinh^4 2\xi_0 + K(-2\cosh^3 2\xi_0 \sinh^2 2\xi_0 + 2\cosh^2 2\xi_0 \sinh^3 2\xi_0)\Big\}$$

$$+ \cos 2\eta \Big\{ -4\cosh^3 2\xi_0 \sinh 2\xi_0 + 6\cosh^2 2\xi_0 \sinh^2 2\xi_0 - 3\cosh 2\xi_0 \sinh^3 2\xi_0$$

$$+ \sinh^4 2\xi_0 + K(\cosh 2\xi_0 \sinh^3 2\xi_0 - \sinh^4 2\xi_0)\Big\} - 2\cosh^2 2\xi_0 \sinh 2\xi_0$$

$$+ \sinh^3 2\xi_0 + K(2\cosh^2 2\xi_0 \sinh 2\xi_0 - \sinh^3 2\xi_0)\Big] \Big/ (\cosh 2\xi_0 - \cos 2\eta) \tag{71}$$

This value of $P^b_{\eta\eta}$ for the matrix may be compared with the value of $p^b_{\eta\eta}$ for the inclusion. It is obvious that

$$P^b_{\eta\eta} \neq p^b_{\eta\eta}$$

The displacement field may be obtained by substituting from (61) and (62) in the last relation of (65). The actual displacement in the inclusion is sum of non-elastic displacements due to temperature-field $b_2 r^2 \cos 2\theta$, and elastic displacements, due to the constraints of the matrix. The displacement in the inclusion is thus :

$$2\mu(u_\xi + i u_\eta) = \frac{K b_2 c^3}{2(K+1)} \left[\frac{K(a^2+b^2)}{6(a+b)^2} \left\{ \cosh 3\xi \cos 3\eta + i \sinh 3\xi \sin 3\eta \right.\right.$$

$$\left. + 3\cosh\xi \cos\eta + 3i \sinh\xi \sin\eta \right\}$$

$$- \frac{(a^2+b^2)}{(a+b)^2} \left(\cosh\xi \cos\eta + i \sinh\xi \sin\eta \right) \left\{ \cosh 2\xi \cos 2\eta + 1 - \right.$$

$$\left. - i \sinh 2\xi \sin 2\eta \right\} - \frac{(K-3)(a^2+b^2)(a-b)}{6(a+b)^3} \left\{ \cosh 3\xi \cos 3\eta \right.$$

$$\left. - i \sinh 3\xi \sin 3\eta + 3\cosh\xi \cos\eta - 3i \sinh\xi \sin\eta \right\} +$$

$$\left. + \frac{4ab}{c^6} \left\{ a^4 + b^4 + 4a^2b^2 - 3a^3b - 3ab^3 + K ab(a-b)^2 \right\} \left(\cosh\xi \cos\eta - i \sinh\xi \sin\eta \right) \right] \tag{72}$$

The boundary value of this displacement is

$$2\mu(u_\xi^b + i u_\eta^b) = \frac{K b_2 c^3}{2(K+1)} \left[\cos 3\eta \left\{ \frac{1}{6} \left(\frac{a}{c} \cosh 2\xi_0 + \frac{b}{c} \sinh 2\xi_0 \right) \times \right.\right.$$

$$\times \left(+ \frac{(a^2+b^2)(a-b)(3-K)}{(a+b)^3} + K \frac{(a^2+b^2)}{(a+b)^2} \right)$$

$$- \frac{a}{2c} \cosh 2\xi_0 \Big\} + i \sin 3\eta \Big\{ \frac{1}{6} \left(\frac{b}{c} \cosh 2\xi_0 + \frac{a}{c} \sinh 2\xi_0 \right) \times$$

$$\times \left(\frac{(K-3)(a^2+b^2)(a-b)}{(a+b)^3} + \frac{K(a^2+b^2)}{(a+b)^2} \right) -$$

$$- \frac{b}{2c} \cosh 2\xi_0 \Big\} + \frac{a}{c} \cos\eta \Big\{ -1 - \frac{1}{2} \left(\frac{(K-3)(a^2+b^2)(a-b)}{(a+b)^3} \right) +$$

$$+ \frac{4ab}{c^6} \left(a^4 + b^4 + 4a^2b^2 - 3a^3b - 3ab^3 + Kab(a-b)^2 \right) +$$

$$+ \frac{K}{2} \frac{(a^2+b^2)}{(a+b)^2} + \frac{2Kab}{(a+b)^2} + \frac{1}{2} \cosh 2\xi_0 \Big\} +$$

$$+ \frac{ib}{c} \sin\eta \Big\{ -1 - \frac{1}{2} \left(\frac{(K-3)(a^2+b^2)(a-b)}{(a+b)^3} \right) -$$

$$- \frac{4ab}{c^6} \left(a^4 + b^4 + 4a^2b^2 - 3a^3b - 3ab^3 + Kab(a-b)^2 \right) +$$

$$+ \frac{K(a^2+b^2)}{2(a+b)^2} + \frac{2Kab}{(a+b)^2} + \frac{1}{2} \cosh 2\xi_0 \Big\} +$$

$$+ \frac{2ab(a^2+b^2)}{(a+b)^2} \left(\frac{a}{c} \cos\eta + i \frac{b}{c} \sin\eta \right) (\cos 2\eta + i \sin 2\eta) \Big]$$

The displacement field in the matrix is given by substituting in the last relation of (66) the values of $\phi'_m(z)$ and $\psi'_m(z)$ from (62) :

$$
\begin{aligned}
2\mu(U_\xi+iU_\eta) = \frac{K b_2 c^3}{2(K+1)}\Bigg[& \frac{K ab(a^2+b^2)}{3c^4}\Big\{-\cosh 3\xi\cos 3\eta - i\sinh 3\xi \sin 3\eta \\
& + \sinh 3\xi\cos 3\eta + i\cosh 3\xi\sin 3\eta - 3\cosh\xi\cos\eta - \\
& - 3i\sinh\xi\sin\eta - 3\sinh\xi\cos\eta - 3i\cosh\xi\sin\eta\Big\} + \frac{2Kab(a^2+b^2)}{c^4}\times \\
& (\cosh\xi\cos\eta + i\sinh\xi\sin\eta) - \frac{2ab(a^2+b^2)}{c^4}\Big\{\sinh 2\xi\cos 2\eta - i\cosh 2\xi\sin 2\eta \\
& - \cosh 2\xi\cos 2\eta + i\sinh 2\xi\sin 2\eta\Big\}(\cosh\xi\cos\eta + i\sinh\xi\sin\eta) \\
& + \frac{2ab(a^2+b^2)^2}{3c^6}(K-3)\Big\{\cos 3\xi\cos 3\eta - i\sinh 3\xi\sin 3\eta - \sinh 3\xi\cos 3\eta \\
& + i\cosh 3\xi\sin 3\eta + 3\cosh\xi\cos\eta - 3i\sinh\xi\sin\eta + 3\sinh\xi\cos\eta \\
& - 3i\cosh\xi\sin\eta\Big\} + \frac{4ab}{c^6}\Big\{(a^2+b^2)^2 - 2a^2b^2(K-1)\Big\}(\cosh\xi\cos\eta - \\
& i\sinh\xi\sin\eta) + \frac{4ab}{c^6}\Big\{2(a^2+b^2)^2 - 2a^2b^2K\big((a^2+b^2)^2 - 2a^2b^2\big)\Big\}\times \\
& (\sinh\xi\cos\eta - i\cosh\xi\sin\eta)\Bigg]
\end{aligned}
\tag{73}
$$

It may be seen that the boundary values of net displacements for inclusion and the displacement of the matrix are continuous.

CHAPTER V

HARMONIC TEMPERATURE DISTRIBUTION IN INFINITE ELASTIC MEDIUM.

Previous work on inclusion problems has been confined to the case where inclusions and inhomogeneities undergo spontaneous homogeneous deformation and the matrix had no such deformation. The matrix simply acted as a constraint to the inclusion which tried to attain its free-state configuration. Here, in this chapter, the problem, when the matrix undergoes spontaneous deformation is considered. But the presence of the inclusion develops stress-field both in the matrix and itself. The explicit solution of a problem forms the subject matter of this chapter.

Consider an infinite elastic medium, with a circular tube, under temperature distribution of the form

$$T(r,\theta) = \frac{b_0 \cos\theta}{r}, \tag{74}$$

with insulated inner boundary, so as not to change the temperature of the inclusion. This type of temperature distribution obviously satisfies steady state heat conduction equation (30).

The problem may be solved directly by the following hypothetical considerations :

Cut out the inclusion. Allow the matrix to undergo the temperature distribution in question. This would attain a prescribed deformation and reduce the size of the cavity from which the inclusion was taken out. Apply surface tractions to the boundary of the cavity to bring it back to the initial shape and size. Fit the inclusion into the cavity and then apply the operations similar to those given in chapter II page 14 .

The final solution must be of the form that it should transmit a perfect bond on the inclusion boundary and the net displacement field is continuous on the boundary.

According to thermo-elastic stress-strain relationship the constrained stress-field in the matrix is

$$P^{\circ}_{xx} = -2\alpha(\lambda+\mu)T, \quad P^{\circ}_{yy} = -2\alpha(\lambda+\mu)T, \quad P^{\circ}_{xy} = 0 \tag{75}$$

where α is the coefficient of linear expansion of the

materials and λ and μ are Lame' constants. Substitution of these values of stresses in (27) provides us with

$$P ds = -2i(\alpha\lambda+\alpha\mu)T d\xi ,$$
$$\bar{P} ds = 2i(\alpha\lambda+\alpha\mu)T d\bar{\xi} . \qquad (76)$$

It may be noted that at the boundary $r = a$,

$$T = \frac{b_0 \cos\theta}{r} = \frac{b_0}{2}\left(\frac{1}{\sigma}+\frac{\sigma}{a^2}\right), \qquad (77)$$

because $\sigma\bar{\sigma} = a^2$ is the boundary of the circle Γ .

The value of T from (77) is substituted in (76) and the values of $P ds$ and $\bar{P} ds$ thus obtained are substituted in (25). The contour integrals are then evaluated. It may be seen that two values of each of $\phi'(z)$ and $\psi'(z)$ are obtained depending upon whether z is out-side or inside the contour Γ . Distinguishing these by subscripts i and m for inclusion and matrix respectively, we get after some calculation

$$\phi'_i(z) = \frac{\alpha(\lambda+\mu)b_0 z}{a^2(K+1)} , \quad \psi'_i(z) = 0 , \qquad (78)$$

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for inclusion; and

$$\phi'_m(z) = -\frac{\alpha(\lambda+\mu)}{(K+1)}\frac{1}{z}$$

$$\psi'_m(z) = \frac{\alpha(\lambda+\mu)}{(K+1)}\left[\frac{K}{z} + \frac{Ka^2}{z^3} - \frac{2a^2}{z^3}\right] \tag{79}$$

for the matrix.

For evaluation of stresses the values of complex-potential functions are substituted from (78) and (79) in (11a) and (11b). Following the procedure outlined above, after some calculations the radial, transverse and tangential stresses in the inclusion would be

$$p_{rr} = \frac{\alpha(\lambda+\mu)}{(K+1)} b_o \cos\theta \left[\frac{r}{a^2}\right]$$

$$p_{\theta\theta} = \frac{\alpha(\lambda+\mu)}{(K+1)} b_o \cos\theta \left[\frac{3r}{a^2}\right] \tag{80}$$

$$p_{r\theta} = \frac{\alpha(\lambda+\mu)}{(K+1)} b_o \sin\theta \left[\frac{r}{a^2}\right]$$

As already stated, at the boundary these components are distinguished by superscript b and have the values

$$p^b_{rr} = \frac{\alpha(\lambda+\mu) b_o \cos\theta}{a(K+1)}$$

$$p_{\theta\theta}^{b} = \frac{3\alpha(\lambda+\mu)\,b_0\cos\theta}{a(K+1)}$$

$$p_{r\theta}^{b} = \frac{\alpha(\lambda+\mu)\,b_0\sin\theta}{a(K+1)} \quad . \tag{81}$$

Now we proceed to find the stress-field P_{rr}, $P_{\theta\theta}$, $P_{r\theta}$ of the matrix. It may be noted that originally the matrix had a stress-field. This is given by 'constrained stress-field' (75). Hence for finding out actual stress-field the constrained stress field is to be superposed upon that obtained by the complex-potentials $\phi_m'(z)$ and $\psi_m'(z)$ given by (79). Thus the radial, transverse and tangential components are

$$P_{rr} = \frac{\alpha(\lambda+\mu)\,b_0\cos\theta}{r(K+1)}\left[K - \frac{a^2}{r^2}(K-1)\right],$$

$$P_{\theta\theta} = \frac{\alpha(\lambda+\mu)\,b_0\cos\theta}{r(K+1)}\left[3K + \frac{a^2}{r^2}(K-1)\right], \tag{82}$$

$$P_{r\theta} = \frac{\alpha(\lambda+\mu)\,b_0\sin\theta}{r(K+1)}\left[K - \frac{a^2}{r^2}(K-1)\right];$$

Boundary values of these stresses are

$$P_{rr}^{b} = \frac{\alpha(\lambda+\mu)\,b_0\cos\theta}{a(K+1)}$$

$$P^{b}_{\theta\theta} = \frac{\alpha(\lambda+\mu)\, b_0 \cos\theta}{a(\kappa+1)}\,(4\kappa-1)$$

$$P^{b}_{r\theta} = \frac{\alpha(\lambda+\mu)\, b_0 \sin\theta}{a(\kappa+1)} \qquad (83)$$

From the expressions (81) and (83) it is observed that the normal and tangential components of stresses are continuous on the equilibrium interface.

The hoop-stresses are discontinuous as expected. The jump in these quantities is

$$P^{b}_{\theta\theta} - p^{b}_{\theta\theta} = -\frac{4\alpha(\lambda+\mu)\, b_0 \cos\theta}{a}\cdot\frac{\kappa-1}{\kappa+1} \qquad (84)$$

Substituting from the equation (78) in (11c), the displacement field in the inclusion can be directly found out. Thus the displacement at any point of the inclusion is given by

$$2\mu(u_x + i u_y) = \frac{\alpha(\lambda+\mu)\, b_0}{2a^2(\kappa+1)}\left[\kappa z^2 - 2r^2\right] \qquad (85)$$

The components of displacements may be transformed to u_r and u_θ in polar coordinates by the identity (13). The boundary value of $u_r + i u_\theta$ (apart from rigid body

motions) is given by

$$2\mu(u_r + i u_\theta) = \frac{\alpha(\lambda+\mu) b_0 e^{i\theta}}{2(K+1)} . K \tag{86}$$

The displacement field in the matrix is found by substituting from equation (79), in (11c). To this the initial displacement field is added. This is done by Hooke's law when the stresses are given by (75). Then the total displacement in the matrix is found to be

$$2\mu(U_x + iU_y) = \frac{\alpha(\lambda+\mu)}{(K+1)}\left[\frac{a^2(K-2)z^2}{2r^4} + \frac{z^2}{r^2}\right] \tag{87}$$

It can be seen that the net displacement of matrix and of inclusion is continuous at the equilibrium boundary.

CHAPTER VI

HALF-PLANE PROBLEM

In this chapter a method to solve the boundary value problems of half-plane has been discussed. This is based on the work of Tiffen ((21)). In this paper, the complex variable method has been combined with Fourier integral approach to find the explicit solutions of some half-plane problems. This technique is simpler and more informative than the other approaches to such problems. For example, Sneddon ((25)) has applied the integral transform technique but the method involves inversion of functions leading to improper integrals, which except in some simpler cases are difficult to evaluate analytically. However, the complex variable approach gives the solutions directly as soon as the potential functions are known.

In the following we shall use $\phi(z)$ and $\psi(z)$ for complex-potential functions, which we have used throughout this thesis instead of the notations $\Omega(z)$ and $\omega(z)$ used by Tiffen ((21)), who used the notations of Stevenson ((5)). However the relation between them is quite simple

$$\phi(z) = \frac{1}{4}\Omega(z) \ , \quad \psi(z) = \frac{1}{4}\omega'(z)$$

It is shown in that paper, that $\psi(z)$ may be expressed in terms of function $\phi(z)$. Thus the boundary value problems of an elastic half-plane are reduced to the determination of one single function $\phi(z)$.

The stresses and displacement are connected with the complex potential functions by the formulae (11). By addition it can be seen that

$$p_{yy} + i p_{xy} = \phi'(z) + \bar{\phi}'(\bar{z}) + \bar{z}\,\phi''(z) + \psi'(z) . \qquad (88)$$

Now, we begin to solve the problem of the half plane. We choose the straight boundary to be real axis, and write for brevity

$$[p_{yy}]_{y=0} = p^{0}_{yy} \ , \quad [p_{xy}]_{y=0} = p^{0}_{xy} \ .$$

(A) Suppose the boundary conditions refer to the tractions on the straight edge. This problem may be solved by solving two simpler problems, namely (i) when the boundary traction consists of the p°_{yy} along with $p^{\circ}_{xy} = 0$, and (ii) when the boundary traction is p°_{xy} with $p^{\circ}_{yy} = 0$. If on the otherhand the boundary traction consists of both p°_{yy} and p°_{xy}, then the result may be obtained by simple superposition. We shall therefore discuss the two simpler problems one by one.

(1) Consider the case when $p^{\circ}_{xy} = 0$ and $p^{\circ}_{yy} \neq 0$.

Let us choose

$$\psi(z) = -z\phi'(z) + \phi(z) \tag{89}$$

Substituting this value in (88), we get

$$p_{yy} + i\,p_{xy} = \phi'(z) + \bar{\phi}'(\bar{z}) - 2iy\,\phi''(z) \tag{90}$$

and, therefore, on the leading edge,

$$p^{\circ}_{yy} = 2\,\mathrm{Re}\{\phi'(z)\}_{y=0} = 2\,\mathrm{Re}\{\phi'(x)\},\quad p^{\circ}_{xy} = 0. \tag{91}$$

It being assumed, in general, that $\lim_{y \to 0} y\,\phi''(z) = 0$ at all points of the real axis. It may be proved that the condition may be relaxed to include those cases, in which this limit exists, but is not zero at a finite number of points of the real axis (though not relevant for the work in this thesis). From (91) it is evident that this combination gives zero shear over the real axis and if we want

$$p^{\circ}_{yy} = f_1(x) \tag{92}$$

We must choose $\phi(z)$ so that

$$2\,\mathrm{Re}\left\{\phi'(x)\right\} = f_1(x) \tag{93}$$

(2) Let us take the next case, when $p^{\circ}_{xy} \neq 0$, $p^{\circ}_{yy} = 0$ and let us choose

$$\psi(z) = -z\,\phi'(z) - \phi(z) \tag{94}$$

Equation (88) yields

$$p_{yy} + i\,p_{xy} = \bar{\phi}'(\bar{z}) - \phi(z) - 2iy\,\phi''(z) \tag{95}$$

Hence on the real axis

$$P^0_{yy} = 0 \;, \; P^0_{xy} = -2\,\mathrm{Im}\{\phi'(x)\}$$

Thus, if

$$P^0_{xy} = f_2(x) \tag{96}$$

on the leading edge, one must choose $\phi(z)$, so that

$$-2\,\mathrm{Im}\{\phi'(x)\} = f_2(x) \tag{97}$$

(B) Next, we consider what is called the second fundamental problem of elasticity theory. Suppose the displacement U^0_x is prescribed on the boundary and $U^0_y = 0$, we choose the function $\psi(z)$ such that

$$\psi(z) = -z\,\phi'(z) - K\,\phi(z) \tag{98}$$

Equation (11c) at once gives

$$2\mu(U_x + iU_y) = K[\phi(z) + \bar{\phi}(\bar{z})] - 2iy\,\bar{\phi}'(\bar{z}) \tag{99}$$

Thus

$$\mu U^0_x = K\,\mathrm{Re}\{\phi(x)\} \;, \quad U^0_y = 0 \tag{100}$$

It being assumed that $\lim_{y \to 0} y\phi'(z) = 0$ at all points of the real axis. This condition may also be relaxed to include those cases where this limit exists but is non-zero or unique at finite number of points of $y = 0$. From (100) it is evident that in this case u_y is zero over the real axis where as u_x is a prescribed function.

If we require

$$u_x^0 = f_3(x) \tag{101}$$

We must choose $\phi(z)$, so that

$$Re\{\phi(x)\} = \frac{\mu}{\kappa} f_3(x) . \tag{102}$$

Finally let $u_y^0 \neq, u_x^0 = 0$ and choose,

$$\psi(z) = -z\phi'(z) + \kappa\phi(z) \tag{103}$$

From (11e)

$$2\mu(u_x + iu_y) = \kappa[\phi(z) - \overline{\Phi}(\bar{z})] - 2iy\,\overline{\Phi}'(\bar{z}) \tag{104}$$

Thus

$$u_x^0 = 0 \qquad ; \quad \mu u_y^0 = \kappa\, Im\{\phi(x)\} \tag{105}$$

Hence to require,

$$u_y^0 = f_4(x) \tag{106}$$

one must choose $\phi(z)$ so that

$$Im\left\{\phi'(x)\right\} = \frac{\mu}{K} f_4(x) \tag{107}$$

The equations (93), (97), (102), (107) reduce the problem of semi-infinite elastic plane $y \geqslant 0$ with specified tractions or displacement along the real axis, to the determination of complex potential functions which are analytic in the upper half plane, and are of suitable orders of magnitude at infinity and have specified real or imaginary part on real axis. In the problems under consideration, the complex potentials at infinity are to be of orders, such that

$$\phi'(z) = O(z^{-1}) \quad , \quad \psi'(z) = O(z^{-1}) \; .$$

From this it is obvious that the stresses at infinity are $O(z^{-1})$. These are the lowest possible orders, if the stresses applied along the real axis have non-zero resultant.

All these cases can be discussed as particular cases if we solve the following problem. Find a function

$F(z)$ which is analytic in the upper half plane and has prescribed real or imaginary part along the real axis.

Let the function $F(z)$ be related to a real function $f(x)$ as follows

$$F(z) = G(x,y) + i\,H(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} Tf(u)\, e^{izu} du$$

where $G(x,y)$ and $H(x,y)$ are real and imaginary parts of $F(z)$ and,

$$Tf(u) = \int_{-\infty}^{\infty} f(t)\, e^{-iut} dt$$

We also assume that the function $f(x)$, is expressible as a Fourier integral, is of bounded variation and at each point

$$f(x) = \left[\tfrac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right]$$

and

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \qquad (108)$$

Subject to (108), $F(z)$ has the properties listed below, (Muskhelishvili ((3))),

(1) $G(x,0) = f(x)$;

(ii) $\lim_{y \to 0^+} G(x, y) = f(x)$,

(iii) $F(z)$ is analytic in $y > 0$, (109)

(iv) $F(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{z-t}$ in $y > 0$,

(v) If in addition to the second condition of (108) it is further assumed that

$$\int_{-\infty}^{\infty} |x f(x)|\,dx < \infty ,$$

$F(z) = O(z^{-1})$ at infinity in upper half plane.

An example is given in appendix A, p. 154 to show the method of evaluation of the function $F(z)$ when $f(x)$ is prescribed.

This example is chosen, because many such integrals will be encountered in the subsequent chapters. Thus the equations (93), (97), (102), (107) are satisfied respectively by choosing

$$\Phi'(z) = \frac{1}{2\pi} \int_{0}^{\infty} T f_1(u)\, e^{izu} du \tag{110}$$

$$\Phi'(z) = -\frac{i}{2\pi} \int_{0}^{\infty} T f_2(u)\, e^{izu} du \tag{111}$$

$$\phi(z) = \frac{\mu}{\kappa\pi}\int_0^\infty T f_3(u)\, e^{izu} du \,, \tag{112}$$

$$\phi(z) = \frac{i\mu}{\kappa\pi}\int_0^\infty T f_4(u)\, e^{izu} du \,; \tag{113}$$

and we have solved all the four problems listed above, namely when

$$p^0_{yy} \neq 0 \quad , \quad p^0_{xy} = 0 \quad ,$$

$$p^0_{yy} = 0 \quad , \quad p^0_{xy} \neq 0 \quad ,$$

$$u^0_x \neq 0 \quad , \quad u^0_y = 0 \quad ,$$

$$u_x = 0 \quad , \quad u^0_y \neq 0 \quad .$$

Having known the values of $\phi(z)$ the corresponding values of $\psi(z)$ can be found from equations (89), (94), (98) and (103). As already remarked the knowledge of $\phi(z)$ and $\psi(z)$ gives the knowledge of elastic field everywhere.

The application of the above theory will be made in two subsequent chapters.

The application of this method enables to solve some problems related to the infinite elastic strip, which are dealt with in chapter X and XI.

CHAPTER VII

CIRCULAR INCLUSION IN ELASTIC HALF-PLANE-I (Traction free edge)

In this chapter, we consider the case of a deforming inclusion in an elastic half-plane, when the leading edge is free from tractions.

Consider a circular region of radius a and centre at a distance ℓ from the leading edge of the half plane. The x-axis is taken along the leading edge, and y-axis is a line perpendicular to the leading edge passing through the centre of the inclusion. The boundary of the inclusion (see fig. page) is given by $(z-i\ell)(\bar{z}+i\ell)=a^2$.

In the absence of matrix the inclusion tends to undergo the displacement characterized by

$$u_x = \delta_1 x + \delta_3 (y-\ell) \quad , \quad u_y = \delta_2 (y-\ell) + \delta_3 x \tag{114}$$

whence the strains are

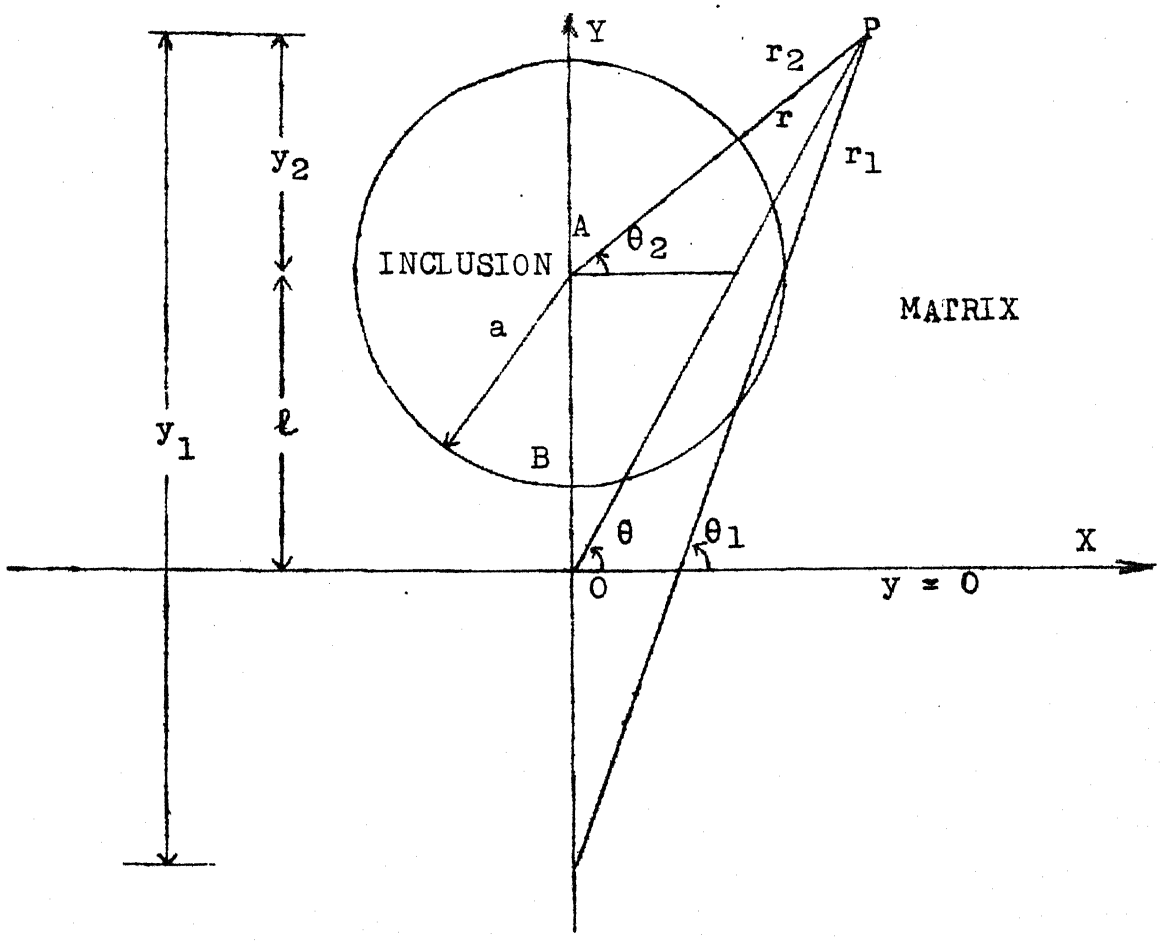

Figure 1, Circular inclusion in semi-infinite medium coordinate system.

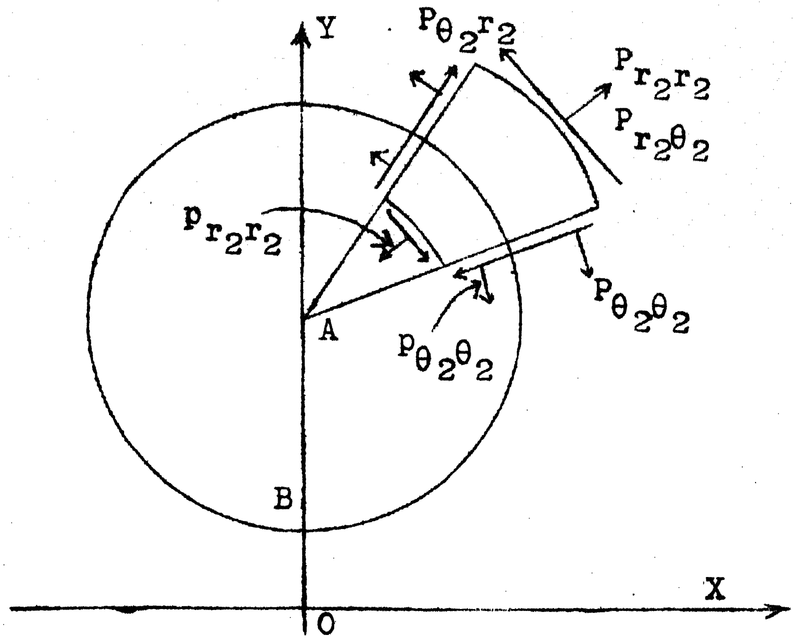

Figure 2, A schematic view of normal and shear stress components in inclusion and matrix.

$$e_{xx} = \delta_1 , \quad e_{yy} = \delta_2 , \quad e_{xy} = \delta_3 . \tag{115}$$

It may be remarked that Bhargava and Kapoor ((15)) solved a similar but simpler problem by using the point-force technique. We use the theory given by Tiffen ((21)) and described in previous chapter. Also the problem is more general in the sense that we take shear strains also into account.

The expression for complex potentials owing to a circular inclusion of radius a , undergoing spontaneous dimensional changes, resulting in the deformation (115) in an infinite elastic medium are given by the complex potentials $\phi_i'(z), \psi_i'(z)$; $\phi_m'(z), \psi_m'(z)$. Then the values are known ((27)) and are given below for ready reference

$$\begin{aligned} \phi_i'(z) &= \frac{-(K-1)}{2(K+1)} (\lambda+\mu)(\delta_1+\delta_2) , \\ \psi_i'(z) &= \frac{\mu}{K+1} (\delta_1-\delta_2-2i\delta_3) ; \end{aligned} \tag{116}$$

$$\begin{aligned} \phi_m'(z) &= -\frac{\mu}{K+1} (\delta_1-\delta_2+2i\delta_3) \frac{a^2}{z^2} , \\ \psi_m'(z) &= \frac{K-1}{K+1} (\lambda+\mu)(\delta_1+\delta_2) \frac{a^2}{z^2} - \frac{\mu}{K+1} (\delta_1-\delta_2+2i\delta_3) \frac{3a^4}{z^4} . \end{aligned} \tag{117}$$

The origin is shifted to $(0, \ell)$. The consequent changes in the complex potentials when the origin is transferred to another point are given by equations (19). In the present case the new complex potentials shall be as follows :

$$\phi_i'(z) = -\frac{K-1}{2(K+1)}(\lambda+\mu)(\delta_1+\delta_2),$$

$$\psi_i'(z) = \frac{\mu}{K+1}(\delta_1-\delta_2-2i\delta_3) \; ; \tag{118}$$

$$\phi_m'(z) = -\frac{\mu}{K+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z_2^2},$$

$$\psi_m'(z) = \frac{K-1}{K+1}(\lambda+\mu)(\delta_1+\delta_2)\frac{a^2}{z_2^2} + \frac{2i\ell\mu}{K+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z_2^3} - \frac{\mu}{K+1}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z_2^4}. \tag{119}$$

where we have retained the same symbols as there is no likelihood of confusion. It may, however, be emphasized again that in these functions, the new origin is the centre of the inclusion.

The stress distribution due to complex potentials (119) is found at the edge $y=0$. Next the stresses P_{yy}^0 and P_{xy}^0 are evaluated at the leading edge. These are nullified by taking additional tractions

p_{yy} and p_{xy} opposite to those found by using (119). Additional complex potentials are now sought for, which superposed on (118), (119) will give the solution of problem under investigation.

Substituting from (119) in (11a) and (11b) and setting $y=0$, we have

$$p^{\circ}_{yy} = -\frac{2\mu(\delta_1-\delta_2)}{K+1}\frac{a^2(x^2-l^2)}{(x^2+l^2)^2} + \frac{8\mu\delta_3}{K+1}\frac{a^2lx}{(x^2+l^2)^2} + \frac{(K-1)}{K+1}\frac{(\lambda+\mu)(\delta_1+\delta_2)a^2(x^2-l^2)}{(x^2+l^2)^2}$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}a^2\left\{\frac{[2(x^2+l^2)-3a^2][(x^2-l^2)^2-4l^2x^2]}{(x^2+l^2)^4}\right\}$$

$$+\frac{8\mu\delta_3}{(K+1)}\frac{a^2lx(x^2-l^2)\{2(x^2+l^2)-3a^2\}}{(x^2+l^2)^4} \equiv f_1(x), \text{ say} \qquad (120)$$

$$p^{\circ}_{xy} = \frac{2(K-1)(\lambda+\mu)(\delta_1+\delta_2)a^2lx}{(K+1)(x^2+l^2)^2} + \frac{4\mu(\delta_1-\delta_2)a^2lx(x^2-l^2)\{2(x^2+l^2)-3a^2\}}{(K+1)(x^2+l^2)^4}$$

$$-\frac{2\mu\delta_3a^2}{K+1}\left\{\frac{[2(x^2+l^2)-3a^2][(x^2-l^2)^2-4l^2x^2]}{(x^2+l^2)^4}\right\} \equiv f_2(x) \text{ say.} \qquad (121)$$

We require to annul these stresses by introduction of complex potentials which have no singularities in upper half plane. For this purpose use is made of the method discussed in the preceding chapter.

From the procedure outlined in previous chapter, the complex potential functions for the two cases

(a) $\quad [p_{yy}]_{y=0} = -P^{0}_{yy} \quad , \quad [p_{xy}]_{y=0} = 0$

(b) $\quad [p_{yy}]_{y=0} = 0 \quad , \quad [p_{xy}]_{y=0} = -P^{0}_{xy}$

will be evaluated seperately and then the resulting complex potential functions will be found by superposition.

These additional complex potentials are found by substituting the values of $f_1(x)$ and $f_2(x)$ from (120) and (121) in equation (110) and (111) respectively. The integral involved therein are solved by the method given in appendix A page 152 . These are

$$\phi_1'(z) = -\frac{a^2(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{(K+1)z_1^2} - \frac{\mu(\delta_1-\delta_2-2i\delta_3)a^2}{K+1}\left[\frac{1}{z_1^2} - \frac{4il}{z_1^3} - \frac{3a^2}{z_1^4}\right] \tag{122}$$

$$\begin{aligned}\psi'(z) = {} & \frac{(K-1)(\lambda+\mu)(\delta_1+\delta_2)a^2}{(K+1)z_1^2} + \frac{\mu(\delta_1-\delta_2-2i\delta_3)a^2}{(K+1)}\left[\frac{2}{z_1^2} - \frac{4il}{z_1^3} - \frac{3a^2}{z_1^4}\right] \\ & + z\left[-\frac{2(K-1)(\lambda+\mu)(\delta_1+\delta_2)a^2}{(K+1)z_1^3} - \frac{2\mu(\delta_1-\delta_2-2i\delta_3)a^2}{(K+1)} \times \right. \\ & \left. \times \left\{\frac{1}{z_1^3} - \frac{6il}{z_1^4} - \frac{6a^2}{z_1^5}\right\}\right]\end{aligned} \tag{123}$$

These are now superposed on (118) and (119) and give the required complex potentials to the problem as follows :

$$\Phi_i'(z) = -\frac{(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{2(K+1)} - \frac{(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{(K+1)}\frac{a^2}{z_1^2}$$

$$- \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\left[\frac{a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right],$$

$$\Psi_i'(z) = \frac{-(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{(K+1)}\frac{a^2}{z_1^2} + \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\left[1+\frac{2a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right] \tag{124}$$

$$+z\left[-\frac{2(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{(K+1)}\frac{a^2}{z_1^3} - \frac{2\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\left\{\frac{a^2}{z_1^3} - \frac{6ila^2}{z_1^4} - \frac{6a^4}{z_1^5}\right\}\right].$$

$$\Phi_m'(z) = -\frac{(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{(K+1)}\frac{a^2}{z_1^2} - \frac{\mu(\delta_1-\delta_2+2i\delta_3)}{(K+1)}\frac{a^2}{z_2^2}$$

$$- \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}\left[\frac{a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right] \tag{125}$$

$$\Psi_m'(z) = \frac{(K-1)(\lambda+\mu)(\delta_1+\delta_2)a^2}{K+1}\left(\frac{1}{z_1^2}+\frac{1}{z_2^2}\right) + \frac{\mu(\delta_1-\delta_2+2i\delta_3)}{K+1}\left(\frac{2ila^2}{z_2^3} - \frac{3a^4}{z_2^4}\right)$$

$$+\mu\frac{(\delta_1-\delta_2-2i\delta_3)}{K+1}\left[\frac{2a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right]$$

$$+z\left[-\frac{2(K-1)(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\frac{a^2}{z_1^3} - \frac{2\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\left\{\frac{a^2}{z_1^3} - \frac{6ila^2}{z_1^4} - \frac{6a^4}{z_1^5}\right\}\right]$$

The stress-field may then be found by substituting these functions in (11a) and (11b).

The stresses in inclusion are given below :

$$p_{xx} = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left\{1+\frac{(x^2-y_1^2)a^2}{r_1^4}-\frac{4yy_1(3x^2-y_1^2)a^2}{r_1^6}\right\}$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-\frac{4(x^2-y_1^2)a^2}{r_1^4}-1+\frac{12ly_1(3x^2-y_1^2)a^2}{r_1^6}+\frac{4yy_1(3x^2-y_1^2)a^2}{r_1^6}\right.$$

$$+\frac{9a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8}+\frac{24yl\{(x^2-y_1^2)^2-4x^2y_1^2\}a^2}{r_1^8}$$

$$\left.-\frac{24a^4y\{y_1(x^2-y_1^2)^2-4x^2y_1^3+4x^2y_1(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$-\frac{2\mu\delta_3}{K+1}\left[-\frac{8xy_1a^2}{r_1^4}-\frac{12lxa^2(x^2-3y_1^2)}{r_1^6}-\frac{8a^2yx(x^2-3y_1^2)}{r_1^6}+\frac{36a^4xy_1(x^2-y_1^2)}{r_1^8}\right.$$

$$\left.+\frac{96lxyy_1a^2(x^2-y_1^2)}{r_1^8}+\frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$p_{yy} = \frac{-(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[1+\frac{(x^2-y_1^2)a^2}{r_1^4}+\frac{4yy_1a^2(3x^2-y_1^2)}{r_1^6}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[1+\frac{4ly_1a^2(3x^2-y_1^2)}{r_1^6}-\frac{4yy_1a^2(3x^2-y_1^2)}{r_1^6}+\frac{3a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8}\right.$$

$$\left.-\frac{24yl\{(x^2-y_1^2)^2-4x^2y_1^2\}a^2}{r_1^8}+\frac{24a^4y\{y_1(x^2-y_1^2)^2-4x^2y_1^3+4x^2y_1(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$-\frac{2\mu\delta_3}{K+1}\left[-\frac{4lx(x^2-3y_1^2)a^2}{r_1^6}+\frac{4yxa^2(x^2-3y_1^2)}{r_1^6}+\frac{12a^4y_1x(x^2-y_1^2)}{r_1^8}\right.$$

$$\left.-\frac{96lxyy_1(x^2-y_1^2)a^2}{r_1^8}-\frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$p_{xy} = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[\frac{2xy_1a^2}{r_1^4} + \frac{4yxa^2(x^2-3y_1^2)}{r_1^6}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-\frac{4xy_1a^2}{r_1^4} - \frac{4lxa^2(x^2-3y_1^2)}{r_1^6} - \frac{4yxa^2(x^2-3y_1^2)}{r_1^6} + \frac{12a^4xy_1(x^2-y_1^2)}{r_1^8}\right.$$

$$\left. + \frac{96yy_1lxa^2(x^2-y_1^2)}{r_1^8} + \frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$-\frac{2\mu\delta_3}{K+1}\left[1+\frac{2(x^2-y_1^2)a^2}{r_1^4} - \frac{4ly_1(3x^2-y_1^2)a^2}{r_1^6} - \frac{4yy_1(3x^2-y_1^2)a^2}{r_1^6} -\right.$$

$$-\frac{24yla^2\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8} - \frac{3a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8}$$

$$\left.+\frac{24a^4y\{y_1(x^2-y_1^2)^2-4x^2y_1^3+4x^2y_1(x^2-y_1^2)\}}{r_1^{10}}\right]$$

where we have used the following notations for brevity

$$y_1 = y+l\,,\quad y_2 = y-l\,;\quad r^2 = x^2+y^2,\ r_1^2 = x^2+y_1^2\,,\ r_2^2 = x^2+y_2^2$$

The stress-field in the matrix P_{xx}, P_{yy} , P_{xy} is also directly obtained by the complex potentials $\phi_m'(z)$ and $\psi_m'(z)$.

$$
\begin{aligned}
P_{xx} = & -\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[\frac{(x^2-y_2^2)a^2}{r_2^4} + \frac{3(x^2-y_1^2)a^2}{r_1^4} - \frac{4yy_1a^2(3x^2-y_1^2)}{r_1^6}\right] \\
& -\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{2(x^2-y_2^2)a^2}{r_2^4} + \frac{2\{(x^2-y_2^2)^2-4x^2y_2^2\}a^2}{r_2^6} - \frac{3a^4\{(x^2-y_2^2)^2-4x^2y_2^2\}}{r_2^8}\right. \\
& + \frac{4(x^2-y_1^2)a^2}{r_1^4} - \frac{4a^2y_1^2(3x^2-y_1^2)}{r_1^6} - \frac{8ly_1a^2(3x^2-y_1^2)}{r_1^6} - \frac{24yla^2\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8} \\
& \left. - \frac{9a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8} + \frac{24a^4y\{4x^2y_1(x^2-y_1^2)+y_1(x^2-y_1^2)^2-4x^2y_1^3\}}{r_1^{10}}\right] \\
& + \frac{2\mu\delta_3}{K+1}\left[-\frac{4xy_2a^2}{r_2^4} - \frac{8xy_2a^2(x^2-y_2^2)}{r_2^6} + \frac{12a^4xy_2(x^2-y_2^2)}{r_2^8} + \frac{8xy_1a^2}{r_1^4}\right. \\
& + \frac{4xy_1a^2(x^2-3y_1^2)}{r_1^6} + \frac{8lxa^2(x^2-3y_1^2)}{r_1^6} - \frac{96lyxy_1a^2(x^2-y_1^2)}{r_1^8} \\
& \left. - \frac{36a^4xy_1(x^2-3y_1^2)}{r_1^8} - \frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]
\end{aligned}
$$

$$
\begin{aligned}
P_{yy} = & \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[\frac{(x^2-y_2^2)a^2}{r_2^4} - \frac{(x^2-y_1^2)a^2}{r_1^4} - \frac{4yy_1(3x^2-y_1^2)a^2}{r_1^6}\right] + \frac{\mu(\delta_1-\delta_2)}{K+1}\left[-\frac{2(x^2-y_2^2)a^2}{r_2^4} +\right. \\
& + \frac{2a^2\{(x^2-y_2^2)^2-4x^2y_2^2\}}{r_2^6} - \frac{3a^4\{(x^2-y_2^2)^2-4x^2y_2^2\}}{r_2^8} - \frac{4y_1y_2a^2(3x^2-y_1^2)}{r_1^6} + \frac{3a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8} \\
& \left. - \frac{24a^2yl\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8} + \frac{24a^4y\{4x^2y_1(x^2-y_1^2)+y_1(x^2-y_1^2)^2-4x^2y_1^3\}}{r_1^{10}}\right] \\
& - \frac{2\mu\delta_3}{K+1}\left[\frac{4xy_2a^2}{r_2^4} - \frac{8xy_2a^2(x^2-y_2^2)}{r_2^6} + \frac{12a^4xy_2(x^2-y_2^2)}{r_2^8} + \frac{4xy_1a^2(x^2-3y_1^2)}{r_1^6}\right. \\
& - \frac{8lxa^2(x^2-y_1^2)}{r_1^6} - \frac{96lxyy_1a^2(x^2-y_1^2)}{r_1^8} + \frac{12a^4xy_1(x^2-y_1^2)}{r_1^8} \\
& \left. + \frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]
\end{aligned}
$$

$$P_{xy} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[-\frac{2xy_2a^2}{r_2^4}-\frac{2xy_1a^2}{r_1^4}-\frac{4yxa^2(x^2-3y_1^2)}{r_1^6}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-\frac{8xy_2a^2(x^2-y_2^2)}{r_2^6}+\frac{12a^4xy_2(x^2-y_2^2)}{r_2^8}-\frac{4xy_1a^2}{r_1^4}\right.$$

$$-\frac{4xya^2(x^2-3y_1^2)}{r_1^6}-\frac{4\ell xa^2(x^2-3y_1^2)}{r_1^6}+\frac{96yy_1\ell xa^2(x^2-y_1^2)}{r_1^8}$$

$$\left.+\frac{12a^4xy_1(x^2-y_1^2)}{r_1^8}+\frac{24a^4y\{x(x^2-y_1^2)^2-4x^3y_1^2-4xy_1^2(x^2-y_1^2)\}}{r_1^{10}}\right]$$

$$-\frac{2\mu\delta_3}{K+1}\left[-\frac{2\{(x^2-y_2^2)^2-4x^2y_2^2\}a^2}{r_2^6}+\frac{3a^4\{(x^2-y_2^2)^2-4x^2y_2^2\}}{r_2^8}+\frac{2(x^2-y_1^2)a^2}{r_1^4}\right.$$

$$-\frac{4yy_1(3x^2-y_1^2)a^2}{r_1^6}-\frac{4\ell y_1a^2(3x^2-y_1^2)}{r_1^6}-\frac{24a^2y\ell\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8}$$

$$\left.-\frac{3a^4\{(x^2-y_1^2)^2-4x^2y_1^2\}}{r_1^8}+\frac{24a^4y\{4x^2y_1(x^2-y_1^2)+y_1(x^2-y_1^2)^2-4x^2y_1^3\}}{r_1^{10}}\right]$$

The hoop stress on the leading edge $y=0$ can be found from the expression $[P_{xx}]_{y=0} = P^0_{xx}$.

The normal and tangential stress transmitted across the bond on the equilibrium boundary are given below : The stress $P_{r_2r_2}$, $P_{\theta_2\theta_2}$, $P_{r_2\theta_2}$ is marked in fig. 2 page 67 ; θ_1, θ_2 are the angles as shown in this figure.

$$P^b_{r_2r_2} = p^b_{r_2r_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(k-1)}{k+1}\left[-1-\frac{2a^2\cos 2\theta_1}{r_1^2}-\frac{a^2\cos(2\theta_1-2\theta_2)}{r_1^2}+\frac{4a^2r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{k+1}\left[-\cos 2\theta_2-\frac{2a^2\cos 2\theta_1}{r_1^2}-\frac{2a^2\cos(2\theta_1-2\theta_2)}{r_1^2}+\frac{8la^2\sin 3\theta_1}{r_1^3}\right.$$

$$+\frac{4la^2\sin(3\theta_1-2\theta_2)}{r_1^3}+\frac{4a^2r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}+\frac{6a^4\cos 4\theta_1}{r_1^4}+\frac{3a^4\cos(4\theta_1-2\theta_2)}{r_1^4}$$

$$\left.+\frac{24a^2lr\sin\theta\cos(4\theta_1-2\theta_2)}{r_1^4}-\frac{24a^4r\sin\theta\sin(5\theta_1-2\theta_2)}{r_1^5}\right]$$

$$-\frac{2\mu\delta_3}{k+1}\left[\sin 2\theta_2-\frac{2a^2\sin 2\theta_1}{r_1^2}-\frac{2a^2\sin(2\theta_1-2\theta_2)}{r_1^2}-\frac{8la^2\cos 3\theta_1}{r_1^3}-\frac{4la^2\cos(3\theta_1-2\theta_2)}{r_1^3}\right.$$

$$-\frac{4a^2r\sin\theta\cos(3\theta_1-2\theta_2)}{r_1^3}+\frac{6a^4\sin 4\theta_1}{r_1^4}+\frac{3a^4\sin(4\theta_1-2\theta_2)}{r_1^4}$$

$$\left.+\frac{24a^2lr\sin(4\theta_1-2\theta_2)}{r_1^4}+\frac{24a^4r\sin\theta\cos(5\theta_1-2\theta_2)}{r_1^5}\right]$$

$$P^b_{r_2\theta_2} = p^b_{r_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(k-1)}{k+1}\left[-\frac{a^2\sin(2\theta_1-2\theta_2)}{r_1^2}-\frac{4a^2r\sin\theta\cos(3\theta_1-2\theta_2)}{r_1^3}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{k+1}\left[\sin 2\theta_2-\frac{2a^2\sin(2\theta_1-2\theta_2)}{r_1^2}-\frac{4la^2\cos(3\theta_1-2\theta_2)}{r_1^3}-\frac{4a^2r\sin\theta\cos(3\theta_1-2\theta_2)}{r_1^3}\right.$$

$$\left.+\frac{3a^4\sin(4\theta_1-2\theta_2)}{r_1^4}+\frac{24a^2lr\sin\theta\sin(4\theta_1-2\theta_2)}{r_1^4}+\frac{24a^4r\sin\theta\cos(5\theta_1-2\theta_2)}{r_1^5}\right]$$

$$-\frac{2\mu\delta_3}{k+1}\left[\cos 2\theta_2+\frac{2a^2\cos(2\theta_1-2\theta_2)}{r_1^2}-\frac{4a^2l\sin(3\theta_1-2\theta_2)}{r_1^3}-\frac{4a^2r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}\right.$$

$$\left.-\frac{3a^4\cos(4\theta_1-2\theta_2)}{r_1^4}-\frac{24a^2lr\sin\theta\cos(4\theta_1-2\theta_2)}{r_1^4}+\frac{24a^4r\sin\theta\sin(5\theta_1-2\theta_2)}{r_1^5}\right]$$

The hoop stress is discontinuous across the boundary. The expressions for hoop-stresses in inclusion and the matrix at the interface are as follows :

$$p^{b}_{\theta_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[-1-\frac{2a^2\cos 2\theta_1}{r_1^2}-\frac{4a^2 r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}+\frac{a^2\cos(2\theta_1-2\theta_2)}{r_1^2}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\cos 2\theta_2-\frac{2a^2\cos 2\theta_1}{r_1^2}+\frac{2a^2\cos(2\theta_1-2\theta_2)}{r_1^2}+\frac{8\ell a^2\sin 3\theta_1}{r_1^3}\right.$$

$$-\frac{4a^2 r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}+\frac{6a^4\cos 4\theta_1}{r_1^4}-\frac{3a^4\cos(4\theta_1-2\theta_2)}{r_1^4}-\frac{24\ell r a^2\sin\theta\cos(4\theta_1-2\theta_2)}{r_1^4}$$

$$\left.+\frac{24a^4 r\sin\theta\sin(5\theta_1-2\theta_2)}{r_1^5}\right]-\frac{2\mu\delta_3}{K+1}\left[-\sin 2\theta_2-\frac{2a^2\sin 2\theta_1}{r_1^2}+\frac{2a^2\sin(2\theta_1-2\theta_2)}{r_1^2}\right.$$

$$-\frac{8a^2\ell\cos 3\theta_1}{r_1^3}+\frac{4a^2\ell\cos(3\theta_1-2\theta_2)}{r_1^3}+\frac{4a^2 r\sin\theta\cos(3\theta_1-2\theta_2)}{r_1^3}+\frac{6a^4\sin 4\theta_1}{r_1^4}$$

$$\left.-\frac{3a^4\sin(4\theta_1-2\theta_2)}{r_1^4}-\frac{24a^2\ell r\sin\theta\sin(4\theta_1-2\theta_2)}{r_1^4}-\frac{24a^4 r\sin\theta\cos(5\theta_1-2\theta_2)}{r_1^5}\right]$$

$$P^{b}_{\theta_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\left[1-\frac{2a^2\cos 2\theta_1}{r_1^2}+\frac{a^2\cos(2\theta_1-2\theta_2)}{r_1^2}-\frac{4a^2 r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-3\cos 2\theta_2-\frac{2a^2\cos 2\theta_1}{r_1^2}+\frac{2a^2\cos(2\theta_1-2\theta_2)}{r_1^2}+\frac{8\ell a^2\sin 3\theta_1}{r_1^3}\right.$$

$$-\frac{4a^2\ell\sin(3\theta_1-2\theta_2)}{r_1^3}-\frac{4a^2 r\sin\theta\sin(3\theta_1-2\theta_2)}{r_1^3}+\frac{6a^4\cos 4\theta_1}{r_1^4}-\frac{3a^4\cos(4\theta_1-2\theta_2)}{r_1^4}$$

$$\left.-\frac{24a^2\ell r\sin\theta\cos(4\theta_1-2\theta_2)}{r_1^4}+\frac{24a^4 r\sin\theta\sin(5\theta_1-2\theta_2)}{r_1^5}\right]-$$

$$-\frac{2\mu\delta_3}{K+1}\Big[3\sin 2\theta_2 - \frac{2a^2\sin 2\theta_1}{r_1^2} + \frac{2a^2\sin(2\theta_1-2\theta_2)}{r_1^2} - \frac{8a^2 l\cos 3\theta_1}{r_1^3} + \frac{4a^2 l\cos(3\theta_1-2\theta_2)}{r_1^3}$$
$$+ \frac{4a^2 r\sin\theta\cos(3\theta_1-2\theta_2)}{r_1^3} +$$
$$+ \frac{6a^4\sin 4\theta_1}{r_1^4} - \frac{3a^4\sin(4\theta_1-2\theta_2)}{r_1^4} - \frac{24a^2 l r\sin\theta\sin(4\theta_1-2\theta_2)}{r_1^4} - \frac{24a^4 r\sin\theta\cos(5\theta_1-2\theta_2)}{r_1^5}\Big]$$

To find the displacement, one needs the expressions for $\phi_i(z)$, $\psi_i(z)$; $\phi_m(z)$ and $\psi_m(z)$ which may be obtained by integrating (124) and (125) where suitable constants signifying rigid body displacements are added. Expressions for them are

$$\phi_i(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)z}{2(K+1)} + \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\frac{a^2}{z_1} +$$
$$+ \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}\left[\frac{a^2}{z_1} - \frac{2ila^2}{z_1^2} - \frac{a^4}{z_1^3}\right]$$

$$\psi_i(z) = \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}z - \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}\frac{a^2}{z_1}$$
$$+ z\left[\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\frac{a^2}{z_1^2} + \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}\left\{\frac{a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right\}\right] \qquad (126)$$

$$\phi_m(z) = \frac{\mu(\delta_1-\delta_2+2i\delta_3)}{K+1}\frac{a^2}{z_2} + \frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\frac{a^2}{z_1} + \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{K+1}\left\{\frac{a^2}{z_1} - \frac{2ila^2}{z_1^2} - \frac{a^4}{z_1^3}\right\}$$

$$\psi_m(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\frac{a^2}{z_2} + \frac{\mu(\delta_1-\delta_2+2i\delta_3)}{K+1}\frac{a^4}{z_2^3} -$$
$$- \frac{il\mu(\delta_1-\delta_2+2i\delta_3)}{K+1}\frac{a^2}{z_2^2} - \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\frac{a^2}{z_1}$$
$$+ z\left[\frac{(\lambda+\mu)(\delta_1+\delta_2)(K-1)}{K+1}\frac{a^2}{z_1^2} + \frac{\mu(\delta_1-\delta_2-2i\delta_3)}{(K+1)}\left\{\frac{a^2}{z_1^2} - \frac{4ila^2}{z_1^3} - \frac{3a^4}{z_1^4}\right\}\right] \qquad (127)$$

We give in the appendix following this chapter the tables containing the values of boundary stresses. Table 1 gives normal (radial) stress for the inclusion and the matrix. It may be remarked that they are the same for both inclusion and the matrix, due to continuity property. Table 2 gives tangential stresses for the inclusion and the matrix, they are again continuous. Table 3 gives hoop stress in the inclusion and Table 4 gives hoop stress in the matrix.

In each table first column gives the angle θ_2 varying from -90^0 to 90^0 with an interval of 30^0. The second column corresponds to the case (i) $\delta_1=\delta_2=\delta$, $\delta_3=0$ whereas third column corresponds to the case (ii) $\delta_1=-\delta_2=\delta$, $\delta_3=0$, and the last column corresponds to the case (iii) $\delta_1=\delta_2=0$, $\delta_3=\delta$. The Poisson's ratio is taken to be equal to 1/3, $K=2$ (plane stress case). The values of ℓ, the distance of the centre of the inclusion from the straight edge, denoted by L in the tables have been taken equal to 8, 6, 4, 2, 1.5, 1.1 .

It is obvious from the tables that the edge effect is confined to a small region around the inclusion and when the distance of inclusion is five to six times the radius of inclusion, the solutions differ slightly from those for the infinite case, the error being of the order of one percent. In the table for $L=1.1$, the sudden change in the values can be marked as we pass from lowest point to some other point. The change is prominent as we approach near and near to the straight edge.

Appendix to Chapter VII

Table 1

θ_2	$\delta_1 = \delta_2 = \delta$, $\delta_3 = 0$	$\delta_1 = -\delta_2 = \delta$, $\delta_3 = 0$	$\delta_1 = \delta_2 = 0$, $\delta_3 = \delta$
		l = [illegible]	
−90	[illegible]	[illegible]	[illegible]
−60	[illegible]	[illegible]	[illegible]5948
−30	[illegible]	[illegible]	[illegible]8079
0	[illegible]	[illegible]	[illegible]1252
30	[illegible]	[illegible]	−[illegible]9151
60	[illegible]	[illegible]	−[illegible]79456
90	[illegible]	[illegible]	[illegible]
		l = 6.	
−90	[illegible]	[illegible]	[illegible]0000
−60	[illegible]	[illegible]	[illegible]5625
−30	[illegible]	[illegible]	[illegible]0989
0	[illegible]	[illegible]	[illegible]2880
30	[illegible]	[illegible]	−[illegible]3426
60	−1.955774	[illegible]	−[illegible]6219
90	[illegible]	.644724	−0.[illegible]0000
		l = 4.	
−90	−1.348106	.382614	[illegible]0000
−60	[illegible]	.291639	[illegible]2037
−30	−1.944341	[illegible]	[illegible]1922
0	−1.905927	[illegible]	0.[illegible]8919
30	−1.955741	[illegible]	−0.[illegible]9304
60	−1.911211	.233941	−0.[illegible]6875
90	−1.813748	.623211	−0.[illegible]000
		l = 2.	
−90	[illegible]	.263374	0.[illegible]0000
−60	−1.671169	[illegible]	0.3[illegible]0122
−30	−1.994272	−[illegible].248793	[illegible].471987
0	−1.942601	−[illegible].628222	0.049816
30	−1.752657	−[illegible].391640	−0.5[illegible]6368
60	−1.581191	[illegible].234778	−0.5[illegible]4881
90	−1.668000	[illegible].552960	−0.000000
		l = 1.5	
−90	−0.500000	−[illegible].041667	−0.000000
−60	−1.521320	[illegible].164553	0.[illegible]8172
−30	−1.987557	−0.171912	0.4354[illegible]9
0	−1.144000	−0.520853	0.071300
30	−1.618842	−0.406772	−0.489718
60	−1.472539	0.203110	−0.529139
90	−1.437500	0.520833	−0.000000
		l = 1.1	
−90	1.240741	−0.117798	0.000000
−60	−1.443187	0.041817	−0.132605
−30	−1.896127	−0.092627	0.391811
0	−1.597723	−0.591637	0.183656
30	−1.292218	−0.410252	−0.474402
60	−1.127554	0.196455	−0.525156
90	−1.096680	0.514778	−0.000000

Table 2

TANGENTIAL STRESS

θ_2	$\delta_1=\delta_2=\delta$, $\delta_3=0$	$\delta_1=-\delta_2=\delta$, $\delta_3=0$	$\delta_1=\delta_2=0$, $\delta_3=\delta$
		L = 8.	
-90	-0.000393	-0.000000	0.655102
-60	-0.019216	-0.564769	0.328980
-30	-0.012017	-0.569340	-0.325936
0	0.013359	-0.002519	-0.656757
30	0.016234	0.569192	-0.330954
60	0.021381	0.569001	0.327803
90	-0.000427	0.000000	0.657565
		L = 6.	
-90	-0.002510	-0.000000	0.645251
-60	-0.027990	-0.557739	0.329275
-30	-0.020119	-0.564070	-0.319036
0	0.007496	-0.006520	-0.640720
30	0.023440	0.561290	-0.329251
60	0.020378	0.565129	0.323452
90	-0.000917	0.000000	0.651299
		L = 4.0	
-90	-0.009831	-0.000000	0.615034
-60	-0.064391	-0.520193	0.321400
-30	-0.056453	-0.552805	-0.296936
0	0.029810	-0.018256	-0.632536
30	0.057037	0.543026	-0.329122
60	0.040959	0.553795	0.311544
90	-0.002743	0.000000	0.635404
		L = 2.0	
-90	-0.074074	-0.000000	0.462905
-60	-0.286536	-0.388115	0.349155
-30	-0.039996	-0.555999	-0.179453
0	0.174232	-0.099255	-0.592240
30	0.206627	0.476327	-0.353874
60	0.119463	0.521439	0.260676
90	-0.016000	0.000000	0.578564
		L = 1.5	
-90	-0.250000	-0.000000	0.375000
-60	-0.443463	-0.307724	0.396772
-30	0.087673	-0.601809	-0.107519
0	0.348060	-0.160960	-0.594720
30	0.325621	0.463548	-0.383430
60	0.169603	0.509160	0.228295
90	-0.031250	0.000000	0.546875
		L = 1.1	
-90	-1.157407	-0.000000	0.422325
-60	-0.188284	-0.258844	0.378529
-30	0.316439	-0.642692	-0.042431
0	0.637986	-0.213663	-0.804459
30	0.480709	0.422952	-0.411549
60	0.223226	0.503703	0.284297
90	-0.061035	0.000039	0.329459

Table 3

HOOP STRESS INSIDE

θ_2	$\delta_1=\delta_2=\delta$, $\delta_3=0$	$\delta_1=-\delta_2=\delta$, $\delta_3=0$	$\delta_1=\delta_2=0$, $\delta_3=0$
		L= 8.0	
-90	-1.998815	-0.661639	-0.000000
-60	-1.988838	-0.335311	-0.567591
-30	-1.972873	0.320256	-0.570617
0	-1.969367	0.651788	-0.003786
30	-1.980445	0.325922	0.567232
60	-1.994804	-0.331925	0.569445
90	-2.000814	-0.661588	0.000000
		L= 6.0	
-90	-1.996995	-0.658011	-0.000000
-60	-1.977642	-0.338235	-0.559813
-30	-1.949774	0.308830	-0.567176
0	-1.946437	0.641209	-0.008760
30	-1.967515	0.321911	0.559154
60	-1.992066	-0.330209	0.564027
90	-2.001821	-0.657795	0.000000
		L= 4.0	
-90	-1.988338	-0.649253	-0.000000
-60	-1.938504	-0.331161	-0.538228
-30	-1.879032	0.274161	-0.563687
0	-1.884863	0.615361	-0.027592
30	-1.936922	0.315071	0.536748
60	-1.987292	-0.324147	0.550715
90	-2.005487	-0.647598	0.000000
		L= 2.0	
-90	-1.831052	-0.658436	-0.000000
-60	-1.571866	-0.484370	-0.476108
-30	-1.471390	0.113635	-0.640011
0	-1.642174	0.556257	-0.152894
30	-1.853607	0.326107	0.446306
60	-1.992502	-0.286574	0.507997
90	-2.032000	-0.604160	0.000000
		L= 1.5	
-90	-1.500000	-0.791667	-0.000000
-60	-0.986682	-0.628962	-0.530377
-30	-1.114484	0.043097	-0.768209
0	-1.516000	0.568587	-0.250560
30	-1.838783	0.360086	0.397378
60	-2.018333	-0.253480	0.489160
90	-2.062500	-0.572917	0.000000
		L= 1.1	
-90	0.314815	-0.982449	-0.000000
-60	0.418916	-0.715230	-0.650494
-30	-0.811759	0.032235	-0.893973
0	-1.506545	0.621681	-0.323670
30	-1.898396	0.412846	0.371503
60	-2.086424	-0.212136	0.482248
90	-2.122070	-0.536140	0.000000

Table 4

HOOP STRESS OUTSIDE

θ_2	$\delta_1=\delta_2=\delta$, $\delta_3=0$	$\delta_1=-\delta_2=\delta$, $\delta_3=0$	$\delta_1=\delta_2=0$, $\delta_3=\delta$
		L= 8.0	
-90	2.001128	2.005028	0.000001
-60	2.011162	0.998022	1.741810
-30	2.027127	-1.013177	1.738784
0	2.030619	-2.014877	-0.003796
30	2.019557	-1.007411	-1.742169
60	2.005196	1.001408	-1.739956
90	1.999186	2.005079	-0.000000
		L= 6.	
-90	2.003005	2.008656	0.000000
-60	2.022198	0.995598	1.749588
-30	2.050216	-1.024504	1.742225
0	2.053563	-2.025458	-0.008760
30	2.032485	-1.011422	-1.750247
60	2.007994	1.003124	-1.749374
90	1.998179	2.008872	-0.000000
		L= 4.0	
-90	1.011662	2.017413	0.000000
-60	2.061496	0.982172	1.771173
-30	2.120998	-1.039172	1.745714
0	2.112117	-2.051306	-0.027592
30	2.063078	-1.018262	-1.772653
60	2.012708	1.009186	-1.758686
90	1.994513	2.019069	-0.000000
		L= 2.0	
-90	2.148148	2.008299	0.000000
-60	2.426034	0.848963	1.833293
-30	2.526650	-1.219698	1.669390
0	2.357826	-2.110410	-0.152894
30	2.146393	-1.007226	-1.863095
60	2.027498	1.046759	-1.801404
90	1.968000	2.062507	-0.000000
		L= 1.5	
-90	2.900000	1.875000	0.000000
-60	3.013318	0.704372	1.779024
-30	2.885916	-1.290236	1.541192
0	2.484000	-2.098080	-0.250560
30	2.161220	-0.973248	-1.912023
60	1.981667	1.079853	-1.820221
90	1.937500	2.093750	-0.000000
		L= 1.1	
-90	4.314815	1.704218	0.000000
-60	4.413916	0.618103	1.658907
-30	3.188241	-1.301099	1.415820
0	2.493451	-2.044986	-0.323670
30	2.101604	-0.920457	-1.937898
60	1.913576	1.121195	-1.827153
90	1.877930	2.130826	-0.000000

CHAPTER VIII

CIRCULAR INCLUSION IN ELASTIC HALF PLANE-II
(Displacement free edge)

In the last chapter, we considered the case of a circular inclusion in a half-plane, when the leading edge is free from stresses. In this chapter, we consider the case when the edge is constrained so that there is no displacement.

To consider this, we have to consider the effect of an isolated force $P = X + iY$ acting at a point ξ, when we have used the same frame of reference as in the last chapter, namely, the leading edge is the x-axis, and y-axis is a line perpendicular to it in the plane. The elastic medium occupies the upper half of the complex plane.

While developing the theory discussed in chapter VI, Tiffen ((21)) has given the complex potential functions $\phi(z)$ and $\psi(z)$ arising due to a point force

P with the straight boundary free from displacements. They were modified to the form suited to our needs and are given below :

$$\phi'(z) = \nu P\left[\frac{1}{\xi - z} + \frac{1}{z - \bar{\xi}}\right] - \frac{\nu \bar{P}(\xi - \bar{\xi})}{\kappa (z - \bar{\xi})^2} ,$$

$$\psi'(z) = -\nu \bar{P}\left[\frac{\kappa}{\xi - z} + \frac{\kappa}{z - \bar{\xi}} + \frac{\xi - \bar{\xi}}{\kappa (z - \bar{\xi})^2} + \frac{2\bar{\xi}(\xi - \bar{\xi})}{\kappa (z - \bar{\xi})^3}\right] \qquad (128)$$

$$+ \nu P\left[\frac{-\bar{\xi}}{(\xi - z)^2} + \frac{\bar{\xi}}{(z - \bar{\xi})^2}\right]$$

where $\bar{P}$ is the complex conjugate of P, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress case. Thus where there is a continuous layer of point forces along a smooth arc Γ of the half-plane, the cumulative effect will be given by integration of the expressions in (128) along the contour Γ , with the respect to the arc length. The expressions will be

$$\phi'(z) = \nu \int_{\Gamma} P\left[\frac{1}{\xi - z} + \frac{1}{z - \bar{\xi}}\right] ds - \nu \int_{\Gamma} \frac{(\xi - \bar{\xi})\bar{P}\, ds}{\kappa (z - \bar{\xi})^2}$$

$$\psi'(z) = -\nu \int_{\Gamma}\left[\frac{\kappa}{\xi - z} - \frac{\kappa}{\bar{\xi} - z} + \frac{\xi - \bar{\xi}}{\kappa (z - \bar{\xi})^2} + \frac{2\bar{\xi}(\xi - \bar{\xi})}{\kappa (z - \bar{\xi})^3}\right]\bar{P}\, ds \qquad (129)$$

$$+ \nu \int_{\Gamma}\left[-\frac{\bar{\xi}}{(\xi - z)^2} + \frac{\bar{\xi}}{(z - \bar{\xi})^2}\right] P\, ds$$

The case of a circular inclusion will now be considered. Inclusion is of radius unity and its centre is at a distance ℓ from the leading edge. Let inclusion be represented by $(z-i\ell)(\bar{z}+i\ell)\leq 1$.

The inclusion in the absence of matrix tends to undergo the displacement characterized by

$$u_x = \delta_1 x + \delta_3(y-\ell) \;,\quad u_y = \delta_2(y-\ell) + \delta_3 x$$

The strain components, therefore, are given by

$$e_{xx} = \delta_1 \;,\; e_{yy} = \delta_2 \quad \text{and} \quad e_{xy} = \delta_3$$

Firstly the case of principal strains $(\delta_3 = 0)$ will be considered. The case of pure shear $(\delta_1 = \delta_2 = 0)$ would be dealt with the latter part of this chapter. If the above deformations are opposed, the stress field generated into the inclusion will be,

$$\begin{aligned} p_{xx} &= -\left\{\lambda(\delta_1+\delta_2)+2\mu\delta_1\right\} \;,\quad p_{xy} = 0 \\ p_{yy} &= -\left\{\lambda(\delta_1+\delta_2)+2\mu\delta_2\right\} \end{aligned} \tag{130}$$

The point force which comes into play on the boundary of the inclusion $(z-i\ell)(\bar{z}+i\ell)=1$ is found

from (129) and (87) and is

$$\bar{P}ds = -i(\lambda+\mu)(\delta_1+\delta_2)d\xi + i\mu(\delta_1-\delta_2)d\bar{\xi}$$

$$\bar{\bar{P}}ds = -i\mu(\delta_1-\delta_2)d\xi + i(\delta_1+\delta_2)(\lambda+\mu)d\bar{\xi} \qquad (131)$$

These expressions are substituted in (129) and the contour integrals are evaluated. It may be noted that on the inclusion boundary Γ, $\bar{\xi} = \frac{1}{\xi - il} - il$, and therefore, $d\bar{\xi} = -d\xi/(\xi-il)^2$.

The expressions will look simpler, if the substitutions $z_1 = z + il = r_1 e^{i\theta_1}$ and $z_2 = z - il = r_2 e^{i\theta_2}$, are made. Thus

$$\phi_i'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[1 + \frac{K-1}{Kz_1^2}\right]$$

$$- \frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{4il}{Kz_1^3} + \frac{3}{Kz_1^4} - \frac{1}{Kz_1^2}\right] \qquad (132)$$

$$\psi_i'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1^2} - \frac{2il(K-1)}{Kz_1^3}\right]$$

$$- \frac{\mu(\delta_1-\delta_2)}{K+1}\left[K + \frac{K}{z_1^2} - \frac{1}{Kz_1^2} + \frac{10il}{Kz_1^3} + \frac{9}{Kz_1^4} + \frac{12l^2}{Kz_1^4} - \frac{12il}{Kz_1^5}\right]$$

$$\Phi'_m(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1^2}\right]$$

$$-\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{4il}{Kz_1^3}+\frac{3}{Kz_1^4}-\frac{1}{Kz_1^2}+\frac{1}{z_2^2}\right]$$

$$\Psi'_m(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1^2}-\frac{2il(K-1)}{Kz_1^3}+\frac{K-1}{z_2^2}\right] \tag{133}$$

$$-\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{K}{z_1^2}-\frac{1}{Kz_1^2}+\frac{10il}{Kz_1^3}+\frac{9}{Kz_1^4}+\frac{12l^2}{Kz_1^4}-\frac{12il}{Kz_1^5}-\frac{2il}{z_2^3}+\frac{3}{z_2^4}\right]$$

The stress field may be found by substituting above complex potential functions in relations (11a) and (11b). But it must be seen that the inclusion has an initial stress field given by (130) and this must be added to the one got from the functions $\Phi'_i(z)$ and $\Psi'_i(z)$. Before proceeding further we can verify that the normal and tangential stresses are continuous across the inclusion boundary. On the leading edge $y=0$, of course the displacement vanishes, as it should. Using the relations

$$p_{r_2r_2} + p_{\theta_2\theta_2} = p_{xx} + p_{yy} ,$$

$$p_{\theta_2\theta_2} - p_{r_2r_2} + 2i\,p_{r_2\theta_2} = (p_{yy} - p_{xx} + 2i\,p_{xy})\, e^{2i\theta_2} ,$$

where $p_{r_2r_2}, p_{r_2\theta_2}, p_{\theta_2\theta_2}$ are radial, transverse and hoop stresses with respect to the centre of circle Γ, (shown in figure 2 page 67). They may be used to evaluate the stress field at any point of the inclusion or the matrix, after superposing the initial stress-field in case of the inclusion, we observe that

$$p_{r_2r_2}+p_{\theta_2\theta_2} = \frac{8(\lambda+\mu)(\delta_1+\delta_2)}{4(K+1)}\left[2+\frac{K-1}{K}\left(\frac{1}{z_1^2}+\frac{1}{\bar{z}_1^2}\right)\right]$$

$$-\frac{8\mu(\delta_1-\delta_2)}{4(K+1)}\left[\frac{4il}{K}\left(\frac{1}{z_1^3}-\frac{1}{\bar{z}_1^3}\right)+\frac{3}{K}\left(\frac{1}{z_1^4}+\frac{1}{\bar{z}_1^4}\right)-\frac{1}{K}\left(\frac{1}{z_1^2}+\frac{1}{\bar{z}_1^2}\right)\right]$$

$$p_{\theta_2\theta_2}-p_{r_2r_2}+2ip_{r_2\theta_2} = \frac{8(\lambda+\mu)(\delta_1+\delta_2)}{4(K+1)}\left[\frac{K-1}{K}\left(-\frac{2z_2}{z_1^3}+\frac{z_2}{\bar{z}_2 z_1^2}\right)\right] \tag{134}$$

$$+\frac{8\mu(\delta_1-\delta_2)}{4(K+1)}\left[\frac{12ilz_2}{Kz_1^4}+\frac{12z_2}{Kz_1^5}-\frac{2z_2}{Kz_1^3}-\frac{Kz_2}{\bar{z}_2}-\frac{Kz_2}{\bar{z}_2 z_1^2}\right.$$

$$\left.+\frac{z_2}{K\bar{z}_2 z_1^2}-\frac{8ilz_2}{K\bar{z}_2 z_1^3}-\frac{9z_2}{K\bar{z}_2 z_1^4}\right]$$

$$p_{r_2r_2}+p_{\theta_2\theta_2} = \frac{8(\lambda+\mu)(\delta_1+\delta_2)}{4(K+1)}\left[\frac{K-1}{K}\left(\frac{1}{z_1^2}+\frac{1}{\bar{z}_1^2}\right)\right]$$

$$-\frac{8\mu(\delta_1-\delta_2)}{4(K+1)}\left[\frac{12ilz_2}{Kz_1^4}+\frac{12z_2}{Kz_1^5}-\frac{2z_2}{Kz_1^3}+\frac{2}{z_2^2}-\frac{Kz_2}{\bar{z}_2 z_1^2}\right. \tag{136}$$

$$\left.+\frac{z_2}{K\bar{z}_2 z_1^2}-\frac{8ilz_2}{K\bar{z}_2 z_1^3}-\frac{9z_2}{K\bar{z}_2 z_1^4}-\frac{3}{\bar{z}_2 z_2^3}\right]$$

$$P_{\theta_2\theta_2} - P_{r_2r_2} + 2i\,P_{r_2\theta_2} = \frac{8(\lambda+\mu)(\delta_1+\delta_2)}{4(K+1)}\left[\frac{K-1}{K}\left(-\frac{2Z_2}{Z_1^3} + \frac{Z_2}{\bar{Z}_2 Z_1^2} + \frac{K}{Z_2\bar{Z}_2}\right)\right]$$

$$+\frac{8\mu(\delta_1-\delta_2)}{4(K+1)}\left[\frac{12ilZ_2}{KZ_1^4} + \frac{12Z_2}{KZ_1^5} - \frac{2Z_2}{KZ_1^3} + \frac{2}{Z_2^2} - \frac{KZ_2}{\bar{Z}_2 Z_1^2}\right.$$

$$\left. + \frac{Z_2}{K\bar{Z}_2 Z_1^2} - \frac{8ilZ_2}{K\bar{Z}_2 Z_1^3} - \frac{9Z_2}{K\bar{Z}_2 Z_1^4} - \frac{3}{\bar{Z}_2 Z_2^3}\right]$$

The normal and tangential stress at the equilibrium interface are given below

$$P^b_{r_2r_2} = P^b_{r_2r_2} = \frac{(\mu+\lambda)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{K}\left\{\frac{2\cos 2\theta_1}{r_1^2} + \frac{2\cos(3\theta_1-\theta_2)}{r_1^3} - \frac{\cos(2\theta_1-2\theta_2)}{r_1^2} - K\right\}\right]$$

$$-\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-\frac{2\cos 2\theta_1}{Kr_1^2} - \frac{\cos(2\theta_1-2\theta_2)}{Kr_1^2} - \frac{K\cos(2\theta_1-2\theta_2)}{r_1^2}\right.$$

$$+\frac{8l\sin 3\theta_1}{Kr_1^3} - \frac{2\cos(3\theta_1-\theta_2)}{Kr_1^3} - \frac{8l\sin(3\theta_1-2\theta_2)}{Kr_1^3}$$

$$+\frac{6\cos 4\theta_1}{Kr_1^4} + \frac{12l\sin(4\theta_1-\theta_2)}{Kr_1^4} - \frac{9\cos(4\theta_1-2\theta_2)}{Kr_1^4}$$

$$\left.+\frac{12\cos(5\theta_1-\theta_2)}{Kr_1^5} + \cos 2\theta_2\right] \qquad (136)$$

$$P^b_{r_2\theta_2} = P^b_{r_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{K}\left\{-\frac{\sin(2\theta_1-2\theta_2)}{r_1^2} + \frac{2\sin(3\theta_1-\theta_2)}{r_1^3}\right\}\right]$$

$$-\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{\sin(2\theta_1-2\theta_2)}{Kr_1^2} - \frac{K\sin(2\theta_1-2\theta_2)}{r_1^2} - \frac{2\sin(3\theta_1-\theta_2)}{Kr_1^3} + \frac{8l\cos(3\theta_1-2\theta_2)}{Kr_1^3}\right.$$

$$\left.-\frac{12l\cos(4\theta_1-\theta_2)}{Kr_1^4} - \frac{9\sin(4\theta_1-2\theta_2)}{Kr_1^4} + \frac{12\sin(5\theta_1-\theta_2)}{Kr_1^5} - \sin 2\theta_2\right] \qquad (137)$$

The hoop stresses are discontinuous across the inclusion boundary and the respective expression for the inclusion and the matrix are ;

$$p^{b}_{\theta_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{K}\left\{\frac{2\cos 2\theta_1}{r_1^2} + \frac{\cos(2\theta_1-2\theta_2)}{r_1^2} - \frac{2\cos(3\theta_1-\theta_2)}{r_1^3}\right\}+2\right]$$

$$+ \frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{\cos(2\theta_1-2\theta_2)}{Kr_1^2} - \frac{K\cos(2\theta_1-2\theta_2)}{r_1^2} + \frac{2\cos 2\theta_1}{Kr_1^2} - \frac{8l\sin 3\theta_1}{Kr_1^3}\right.$$

$$- \frac{8l\sin(3\theta_1-2\theta_2)}{Kr_1^3} - \frac{2\cos(3\theta_1-\theta_2)}{Kr_1^3} - \frac{6\cos 4\theta_1}{Kr_1^4} + \frac{12l\sin(4\theta_1-\theta_2)}{Kr_1^4}$$

$$\left. - \frac{9\cos(4\theta_1-2\theta_2)}{Kr_1^4} + \frac{12\cos(5\theta_1-\theta_2)}{Kr_1^5} + \cos 2\theta_2\right]$$

$$p^{b}_{\theta_2\theta_2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{K}\left\{\frac{2\cos 2\theta_1}{r_1^2} + \frac{\cos(2\theta_1-2\theta_2)}{r_1^2} - \frac{2\cos(3\theta_1-\theta_2)}{r_1^3}\right\}+K-1\right] \quad (138)$$

$$+ \frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{\cos(2\theta_1-2\theta_2)}{Kr_1^2} - \frac{K\cos(2\theta_1-2\theta_2)}{r_1^2} + \frac{2\cos 2\theta_1}{Kr_1^2} - \frac{8l\sin 3\theta_1}{Kr_1^3}\right.$$

$$- \frac{8l\sin(3\theta_1-2\theta_2)}{Kr_1^3} - \frac{2\cos(3\theta_1-\theta_2)}{Kr_1^3} - \frac{6\cos 4\theta_1}{Kr_1^4} + \frac{12l\sin(4\theta_1-\theta_2)}{Kr_1^4}$$

$$\left. - \frac{9\cos(4\theta_1-2\theta_2)}{Kr_1^4} + \frac{12\cos(5\theta_1-\theta_2)}{Kr_1^5} - 3\cos 2\theta_2\right]$$

If once the displacement field in the matrix or that in the inclusion are known the equilibrium boundary

may be found.

Integrating the expressions (132) and (133), we get the following expression to be used in evaluating displacement.

$$\phi_i(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[z_2 - \frac{K-1}{Kz_1}\right] + \frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{2il}{Kz_1^2} + \frac{1}{Kz_1^3} - \frac{1}{Kz_1}\right]$$

$$\psi_i(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1} - \frac{il(K-1)}{Kz_1^2}\right] +$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[-Kz_2 + \frac{K}{z_1} - \frac{1}{Kz_1} + \frac{5il}{Kz_1^2} + \frac{3}{Kz_1^3} + \frac{4l^2}{Kz_1^3} - \frac{3il}{Kz_1^4}\right]$$

$$\phi_m(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1}\right] + \frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{2il}{Kz_1^2} + \frac{1}{Kz_1^3} - \frac{1}{Kz_1} + \frac{1}{z_2}\right]$$

$$\psi_m(z) = -\frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[\frac{K-1}{Kz_1} - \frac{il(K-1)}{Kz_1^2} + \frac{K-1}{z_2}\right] +$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[\frac{K}{z_1} - \frac{1}{Kz_1} + \frac{5il}{Kz_1^2} + \frac{3}{Kz_1^3} + \frac{4l^2}{Kz_1^3} - \frac{3il}{Kz_1^4} - \frac{il}{z_2^2} + \frac{1}{z_2^3}\right]$$

By making use of above expression, the displacement fields in the inclusion and the matrix are given by

$$2\mu(u_x + iu_y) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{K+1}\left[(K-1)z_2 - \frac{K-1}{z_1} - il - \frac{(K-1)z_2}{K\bar{z}_1^2} + \frac{K-1}{K\bar{z}_1}\right] +$$

$$+\frac{\mu(\delta_1-\delta_2)}{K+1}\left[K\bar{z}_2 - \frac{1}{z_1} + \frac{2il}{z_1^2} + \frac{1}{z_1^3} - \frac{K}{\bar{z}_1} + \frac{1}{K\bar{z}_1} - \frac{z_2}{K\bar{z}_1^2} +\right.$$

$$\left.+\frac{4il}{K\bar{z}_1^2} - \frac{3}{K\bar{z}_1^3} - \frac{4ilz_2}{K\bar{z}_1^3} + \frac{3z_2}{K\bar{z}_1^4}\right]$$

and

$$2\mu(U_x+iU_y) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{k+1}\left[(k-1)\left\{-\frac{1}{z_1}-\frac{1}{k\bar{z}_1}-\frac{z_2}{k\bar{z}_1^2}+\frac{1}{\bar{z}_2}\right\}\right]$$

$$+\frac{\mu(\delta_1-\delta_2)}{k+1}\left[-\frac{1}{z}+\frac{k}{z_2}-\frac{k}{\bar{z}_1}+\frac{1}{k\bar{z}_1}+\frac{2il}{z_1^2}-\frac{z_2}{k\bar{z}_1^2}+\frac{4il}{k\bar{z}_1^2}\right.$$

$$\left.+\frac{1}{z_1^3}-\frac{4ilz_2}{k\bar{z}_1^3}-\frac{3}{k\bar{z}_1^3}+\frac{1}{\bar{z}_2^3}+\frac{3z_2}{k\bar{z}_1^4}\right]$$

The case of pure shear can be dealt with in a similar fashion. In this case we have $\delta_1 = \delta_2 = 0$ and $\delta_3 \neq 0$. The relevant complex potentials in this case are :

$$\phi_i'(z) = \frac{8i\mu\delta_3}{4(k+1)}\left[-\frac{1}{kz_1^2}+\frac{4il}{kz_1^3}+\frac{3}{kz_1^4}\right]$$

$$\psi_i'(z) = \frac{8i\mu\delta_3}{4(k+1)}\left[k+\frac{k}{z_1^2}-\frac{1}{kz_1^2}+\frac{10il}{kz_1^3}+\frac{12l^2+9}{kz_1^4}-\frac{12il}{kz_1^5}\right] \tag{139}$$

$$\phi_m'(z) = \frac{8i\mu\delta_3}{4(k+1)}\left[-\frac{1}{kz_1^2}+\frac{4il}{kz_1^3}+\frac{3}{kz_1^4}-\frac{1}{z_2^2}\right]$$

$$\psi_m'(z) = \frac{8i\mu\delta_3}{4(k+1)}\left[\frac{k}{z_1^2}-\frac{1}{kz_1^2}+\frac{10il}{kz_1^3}+\frac{12l^2+9}{kz_1^4}-\frac{12il}{kz_1^5}+\frac{2il}{z_2^3}-\frac{3}{z_2^4}\right] \tag{140}$$

Now the stresses can be found by substituting these expressions in (11a) and (11b) and noting that the initial stress-field is also to be superposed in case of the inclusion. As regards displacement, the expressions (139) and (140) will have to be integrated. The results of integration are as follows

$$\phi_i(z) = -\frac{8i\mu\delta_3}{4(K+1)}\left[-\frac{1}{Kz_1} + \frac{2il}{Kz_1^2} + \frac{1}{Kz_1^3}\right]$$

$$\psi_i(z) = -\frac{8i\mu\delta_3}{4(K+1)}\left[-Kz_2 + \frac{K}{z_1} - \frac{1}{Kz_1} + \frac{5il}{Kz_1^2} + \frac{4l^2+3}{Kz_1^3} - \frac{3il}{Kz_1^4}\right]$$

$$\phi_m(z) = -\frac{8i\mu\delta_3}{4(K+1)}\left[-\frac{1}{z_2} - \frac{1}{Kz_1} + \frac{2il}{Kz_1^2} + \frac{1}{Kz_1^3}\right]$$

$$\psi_m(z) = -\frac{8i\mu\delta_3}{4(K+1)}\left[\frac{K}{z_1} - \frac{1}{Kz_1} + \frac{5il}{Kz_1^2} + \frac{4l^2+3}{Kz_1^3} - \frac{3il}{Kz_1^4} + \frac{il}{z_2^2} - \frac{1}{z_2^3}\right]$$

The displacements in inclusion and matrix are found by substituting expressions for $\phi_i(z), \psi_i(z), \phi_i'(z), \psi_i'(z);$

$\phi_m(z), \psi_m(z)$, $\phi_m'(z), \psi_m'(z)$ in (11c).

In the appendix following this chapter the values of the resultant boundary stresses are given in form of tables in the manner shown in preceding chapter, i.e. first table gives normal (radial) stress for inclusion and matrix, second table gives tangential stress for the inclusion and the matrix. They are the same for inclusion and matrix due to continuity property. Third table gives hoop stress in the inclusion and fourth table gives hoop stress for the matrix.

The first column gives θ_2, second column gives the stresses for the case (i) $\delta_1=\delta_2=\delta$, $\delta_3=0$, and the last column corresponds to the case (ii) $\delta_1=-\delta_2=\delta$, $\delta_3=0$. As in the preceding chapter, ν has been taken equal to 1/3 and $\kappa=2$ (the plane stress case), and ℓ (denoted as L in the tables) takes the values 8, 6, 4, 2, 1.5, 1.1 . The stresses at points of the interface, which are near to the straight edge are of particular importance. In the tables for $L=1.1$, the sudden change in the values can be marked as we pass from the lowest point to some other point. The change is prominent as we approach near and near to the straight edge.

Appendix to Chapter VIII

Table 1

NORMAL STRESS

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L= 8.0	
-90	-2.01274070	0.68026206
-60	-2.01016229	0.34062188
-30	-2.00559860	-0.33665457
0	-2.00395110	-0.67209818
30	-2.00607547	-0.33409570
60	-2.00932300	0.34009286
90	-2.01078767	0.67668927
	L= 6.0	
-90	-2.02329072	0.69200853
-60	-2.01827750	0.34622599
-30	-2.00979280	-0.34007065
0	-2.00708416	-0.67768539
30	-2.01090232	-0.33408784
60	-2.01630005	0.34503495
90	-2.01866177	0.68355539
	L= 4.0	
-90	-2.05539355	0.72910096
-60	-2.04198015	0.36149989
-30	-2.02133447	-0.35214866
0	-2.01630223	-0.69003579
30	-2.02488062	-0.33273011
60	-2.03541592	0.35806446
90	-2.03978050	0.70074006
	L= 2.0	
-90	-2.25925922	0.97942385
-60	-2.17523953	0.41745408
-30	-2.07965404	-0.44244707
0	-2.07103601	-0.73779166
30	-2.10009715	-0.31598276
60	-2.12634450	0.41043044
90	-2.13599998	0.76351999
	L= 1.5	
-90	-2.50000000	1.27083331
-60	-2.31983086	0.43087535
-30	-2.14139938	-0.54668689
0	-2.13199997	-0.76957332
30	-2.17364582	-0.29957248
60	-2.20704320	0.44476422
90	-2.21875000	0.80208333
	L= 1.1	
-90	-2.92592591	1.75334358
-60	-2.66768003	0.98428008
-30	-2.27358067	-0.74690222
0	-2.24835032	-0.81677276
30	-2.30155247	-0.28280736
60	-2.34080470	0.47040923
90	-2.35400391	0.84026718

Table 2

TANGENTIAL STRESS

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L= 8.0	
-90	0.00000000	-0.00000000
-60	0.00357676	-0.58655342
-30	0.00283898	-0.58469128
0	-0.00096144	0.00174410
30	-0.00369108	0.58527613
60	-0.00310768	0.58308245
90	-0.00000000	0.00000000
	L= 6.0	
-90	0.00000000	-0.00000000
-60	0.00642096	-0.59447057
-30	0.00462861	-0.58986074
0	-0.00225151	0.00405142
30	-0.00665942	0.59128277
60	-0.00534492	0.58843791
90	-0.00000000	0.00000000
	L= 4.0	
-90	0.00000000	-0.00000000
-60	0.01447047	-0.61909048
-30	0.00822102	-0.60241757
0	-0.00734092	0.01291474
30	-0.01516926	0.60751326
60	-0.01119144	0.59932765
90	-0.00000000	0.00000000
	L= 2.0	
-90	0.00000000	-0.00000000
-60	0.04457270	-0.76913150
-30	-0.00078836	-0.62902120
0	-0.04884999	0.07730919
30	-0.05666899	0.67498754
60	-0.03485612	0.63649290
90	-0.00000000	0.00000000
	L= 1.5	
-90	0.00000000	-0.00000000
-60	0.02975597	-0.91991282
-30	-0.04292253	-0.61814299
0	-0.09600000	0.14047999
30	-0.09026848	0.72111753
60	-0.05137970	0.65818946
90	-0.00000000	0.00000000
	L= 1.1	
-90	-0.00000000	-0.00000000
-60	-0.20794070	-1.22013047
-30	-0.15253137	-0.57343893
0	-0.16965840	0.23584249
30	-0.13394940	0.77834065
60	-0.07120556	0.68280306
90	-0.00000000	0.00000000

Table 3

HOOP STRESS INSIDE

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L= 8.	
-90	3.99496291	-0.67362908
-60	3.99379434	-0.33427095
-30	3.9891[illegible]429	0.34233400
0	3.9885[illegible]8[illegible]2	0.67756509
30	3.9915[illegible]394	0.33272822
60	3.9952985[illegible]	-0.33597[illegible]47
90	3.98694681	-0.67256630
	L= 6.	
-90	3.99[illegible]23286	-0.67925894
-60	3.9862[illegible]408	-0.33426643
-30	3.98[illegible][illegible]5679	0.35027790
0	3.97987843	0.68519712
30	3.985668[illegible][illegible]	0.34143805
60	3.99224511	-0.33830760
90	3.99499312	-0.67701967
	L= 4.	
-90	3.97376087	-0.69578151
-60	3.96453367	-0.33173824
-30	3.953[illegible][illegible]579	0.37492319
0	3.98665723	0.70723078
30	3.971212[illegible]0	0.34678028
60	3.98501182	-0.34546152
90	3.99039778	-0.68654679
	L= 2.	
-90	3.81481478	-0.78189299
-60	3.79785680	-0.27823858
-30	3.80746469	0.50827109
0	3.86342359	0.77377462
30	3.92322630	0.34374096
60	3.96267083	-0.38453207
90	3.97598995	-0.73791999
	L= 1.5	
-90	3.50000000	-0.85416665
-60	3.57383159	-0.19868597
-30	3.69241977	0.61110448
0	3.81199995	0.79570665
30	3.90145645	0.32291434
60	3.95247954	-0.41957896
90	3.96875000	-0.77604166
	L= 1.1	
-90	2.148148[illegible]0	-1.21322018
-60	3.18231541	-0.84757952
-30	3.62752342	0.75709836
0	3.79798439	0.80175099
30	3.89685959	0.28151069
60	3.94779369	-0.47156783
90	3.96337888	-0.82958642

Table 4

HOOP STRESS OUTSIDE

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L= 8.0	
-90	1.99496293	1.99304163
-60	1.99275497	0.99906237
-30	1.98910441	-0.99099933
0	1.98850805	-1.98859994
30	1.99150398	-0.99480500
60	1.99529852	0.99736279
90	1.99694684	1.99400032
	L= 6.0	
-90	1.99023288	1.98740770
-60	1.98620900	0.99906687
-30	1.98005682	-0.98309602
0	1.97987846	-1.98046951
30	1.98566803	-0.99189527
60	1.99224514	0.99502565
90	1.99499315	1.98964697
	L= 4.0	
-90	1.97376090	1.97088513
-60	1.96453361	1.00159508
-30	1.95300575	-0.95841012
0	1.95665725	-1.95943585
30	1.97121201	-0.98655304
60	1.98501183	0.98787174
90	1.99039780	1.97811989
	L= 2.0	
-90	1.81481479	1.88477363
-60	1.79780681	1.05509473
-30	1.80746469	-0.82506223
0	1.86342354	-1.89289202
30	1.92322642	-0.98958436
60	1.96267085	0.94880118
90	1.97599998	1.92874664
	L= 1.5	
-90	1.50000000	1.81249999
-60	1.57383151	1.13464735
-30	1.69241980	-0.72222883
0	1.81199998	-1.87095997
30	1.90145648	-1.01041898
60	1.95247956	0.91375429
90	1.96875000	1.89062499
	L= 1.1	
-90	0.14814810	1.45344643
-60	1.18231542	1.28575981
-30	1.62752345	-0.57623495
0	1.79798441	-1.86491564
30	1.89685951	-1.05182283
60	1.94779371	0.86176543
90	1.96337889	1.83708063

CHAPTER IX

A POINT-FORCE IN AN INFINITE ELASTIC STRIP.

In this section a brief review of the work done by Tiffen ((22)) is given. This relates to finding the complex potentials for the case of an isolated force in the interior of infinite strip, when

(i) both the straight boundaries are free from stresses ;

(ii) both the straight boundaries are free from displacements.

Previous investigations relating to elastic strips in a state of generalized plane stress were made by Filon ((28)) Howland ((29)), Hopkins ((30)), Sneddon ((25, 26)) and others. However in these problems there was no force in the interior of the strip. Only the boundary tractions or displacements were involved. Moreover the real variable technique was used, which is a bit laborious. But Tiffen gave the powerful

technique of using the complex variables for solving such problems.

The elastic material occupies the region defined by

$$-\infty \leq x \leq \infty \quad , \quad 0 \leq y \leq c_0 \ , \tag{141}$$

where c_0 is constant. The boundary stresses and displacements are denoted by

$$[p_{yy}]_{y=0} = p^0_{yy} \quad , \quad [p_{yy}]_{y=c_0} = p^1_{yy}$$

$$[u_x]_{y=0} = u^0_x \quad , \quad [u_x]_{y=c_0} = u^1_x$$

Firstly consider the problem of a strip subjected to the following boundary tractions :

$$\begin{aligned} p^0_{yy} + i p^0_{xy} &= f(x) + i F(x) \\ p^1_{yy} + i p^1_{xy} &= g(x) + i G(x) \end{aligned} \tag{142}$$

Now our aim is to find out the complex potentials due to a point force in the interior of the strip. As shown by Tiffen, the following operations are performed. First we assume that $f(x)$, $F(x)$, $g(x)$, $G(x)$, satisfy sufficient conditions for existence of Fourier transforms which is found from the following relations

$$f_T(u) = \alpha_1(u) + i\alpha_2(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux} dx ,$$

$$F_T(u) = \epsilon_1(u) + i\epsilon_2(u) = \int_{-\infty}^{\infty} F(x)\, e^{-iux} dx ;$$

$$g_T(u) = \sigma_1(u) + i\sigma_2(u) = \int_{-\infty}^{\infty} g(x)\, e^{-iux} dx ,$$

$$G_T(u) = \tau_1(u) + i\tau_2(u) = \int_{-\infty}^{\infty} G(x)\, e^{-iux} dx , \tag{143}$$

for all real values of u , $\alpha_1(u)$, $\alpha_2(u)$, $\epsilon_1(u)$, $\epsilon_2(u)$, $\sigma_1(u)$, $\sigma_2(u)$, $\tau_1(u)$, $\tau_2(u)$ are bounded for all non negative values of parameter u . For the sake of brevity following notations are used

$$t = uc_0, \quad s = \sinh t , \quad c = \cosh t$$

The real integrable functions $\beta_1(u)$, $\beta_2(u)$, $\gamma_1(u)$, $\gamma_2(u)$ are known from the following relations :

$$\beta_1 = \frac{1}{s^2 - t^2}\left[\alpha_2\,(t^2 - s^2 + t + cs) - \sigma_2\,(ct + s) - t\,(t\,\epsilon_1 + s\,\tau_1)\right],$$

$$\beta_2 = \frac{1}{s^2 - t^2}\left[\alpha_1\,(s^2 - t^2 - t - cs) + \sigma_1\,(tc + s) - t\,(t\,\epsilon_2 + s\,\tau_2)\right];$$

$$\gamma_1 = \frac{1}{s^2 - t^2}\left[\epsilon_2\,(t^2 - s^2 + cs - t) + \tau_2\,(tc - s) + t\,(t\alpha_1 - s\sigma_1)\right],$$

$$\gamma_2 = \frac{1}{s^2 - t^2}\left[\epsilon_1\,(s^2 - t^2 + t - cs) + \tau_1\,(s - tc) + t\,(t\alpha_2 - s\sigma_2)\right]. \tag{144}$$

Then $I(z)$, $J(z)$, $H(z)$ and $K(z)$ are found as follows :

$$I(z) = \frac{1}{2\pi}\int_0^{\infty}\{\alpha_1(u)+i\alpha_2(u)\}\, e^{izu}\,du$$

$$J(z) = -\frac{i}{2\pi}\int_0^{\infty}(\epsilon_1(u)+i\epsilon_2(u))\, e^{izu}\,du$$

$$H(z) = \frac{i}{4\pi}\int_0^{\infty}\left[(\beta_1+i\beta_2)\, e^{izu}+(\beta_1-i\beta_2)\, e^{-izu}\right]du \tag{145}$$

$$K(z) = \frac{1}{4\pi}\int_0^{\infty}\left[(\gamma_1+i\gamma_2)\, e^{izu}+(\gamma_1-i\gamma_2)\, e^{-izu}\right]du$$

from which $\phi_r(z)$ and $\psi_r(z)$ $r=0,1,2,3$ are found as follows :

$$\phi_0(z) = \int I(z)dz\ , \quad \psi_0(z) = -z\phi_0'(z)+\phi_0(z)\ ,$$

$$\phi_1(z) = \int J(z)dz\ , \quad \psi_1(z) = -z\phi_1'(z)-\phi_1(z)\ ,$$

$$\phi_2(z) = \int H(z)dz\ , \quad \psi_2(z) = -z\phi_2'(z)+\phi_2(z)\ , \tag{146}$$

$$\phi_3(z) = \int K(z)dz\ , \quad \psi_3(z) = -z\phi_3'(z)-\phi_3(z)\ ,$$

and hence

$$\phi(z) = \phi_0(z)+\phi_1(z)+\phi_2(z)+\phi_3(z) \tag{147}$$

$$\psi(z) = \psi_0(z)+\psi_1(z)+\psi_2(z)+\psi_3(z)$$

As shown by Tiffen these potential functions solve the problem of an infinite elastic strip subjected to the specified tractions on the straight edges.

The effect of point force in an infinite strip may now be easily found as follows :

Consider an infinite plate in the (x, y) plane where a force $P = X+iY$ acts at point $z = b+ia$ $(0<a<c)$.

This gives rise to stresses everywhere in infinite plate. Suppose we consider the tractions transmitted on an infinite elastic strip (141) cut off from the infinite plate. We nullify these tractions by applying tractions opposite to those transmitted by the infinite plate. We superpose these on the stresses already present in the strip. This gives us stress-field in the infinite strip.

The complex potentials, due to an isolated force $P=X+iY$ at any point $z = b+ia$, in an infinite medium are given with the use of (34) as

$$\phi(z) = -\frac{P \operatorname{Log}(z-b-ia)}{2\pi(K+1)}$$

$$\psi(z) = \frac{K\bar{P}\log(z-b-ia)}{2\pi(K+1)} + \frac{(b-ia)P}{2\pi(K+1)}\frac{1}{(z-b-ia)} \tag{148}$$

The stresses at any point (x, y) on the infinite medium are given by

$$2\pi(K+1)\,p_{yy} = \frac{(K-1)\left[X(x-b)-Y(y-a)\right]}{(x-b)^2+(y-a)^2} - \frac{4(y-a)^2\left[X(x-b)+Y(y-a)\right]}{\left\{(x-b)^2+(y-a)^2\right\}^2},$$

$$2\pi(K+1)\,p_{xy} = -\frac{(K-1)Y(x-b)+(K+3)X(y-a)}{(x-b)^2+(y-a)^2} - \frac{4(y-a)^2\left[Y(x-b)-X(y-a)\right]}{\left\{(x-b)^2+(y-a)^2\right\}^2}. \qquad (149)$$

If the boundary of the strip is to be stress-free, we must nullify the tractions on its straight edges. This is obtained by applying surface tractions on the boundary which are opposite to those found from equations (149). More explicitly we should apply the following tractions on the boundary of the strip

$$2\pi(K+1)\,p^0_{yy} = -\frac{(K-1)\left[X(x-b)+Y(+a)\right]}{(x-b)^2+a^2} + \frac{4a^2\left[X(x-b)-Ya\right]}{\left\{(x-b)^2+a^2\right\}^2}$$

$$2\pi(K+1)\,p^0_{xy} = \frac{(K-1)Y(x-b)-(K+3)X.a}{(x-b)^2+a^2} + \frac{4a^2\left[Y(x-b)+X.a\right]}{\left\{(x-b)^2+a^2\right\}^2}$$

$$2\pi(K+1)\,p^1_{yy} = \frac{-(K-1)\left[X(x-b)-Y(c_0-a)\right]}{(x-b)^2+(c_0-a)^2} + \frac{4(c_0-a)^2\left[X(x-b)+Y(c_0-a)\right]}{\left\{(x-b)^2+(c_0-a)^2\right\}^2} \qquad (150)$$

$$2\pi(K+1)\,p^1_{xy} = \frac{(K-1)Y(x-b)+(K+3)X(c_0-a)}{(x-b)^2+(c_0-a)^2} + \frac{4(c_0-a)^2\left[Y(x-b)-X(c_0-a)\right]}{\left[(x-b)^2+(c_0-a)^2\right]^2}$$

Note that these have been obtained by putting $y=0$ and $y=c_0$ in equations (149), and changing the sign

of the terms.

Hence we have to solve another problem, when on the edges of the strip, tractions given by equations (150) are applied. This is done with the help of the results given earlier in this chapter. With these values of p_{yy}^{0} , p_{xy}^{0} , p_{yy}^{1} and p_{xy}^{1} the functional values of $f(x)$, $F(x)$, $g(x)$, $G(x)$ are determined from equation (142). These are then substituted in (143) and $\alpha_1(u)+i\alpha_2(u)$, $\epsilon_1(u)+i\epsilon_2(u)$, $\sigma_1(u)+i\sigma_2(u)$, $\tau_1(u)+i\tau_2(u)$

evaluated. These values are as follows

$$\alpha_1(u)+i\alpha_2(u)=\frac{i\,e^{-iu(b-ia)}}{2(K+1)}\left[K P-(1+2au)\bar{P}\right],$$

$$\epsilon_1(u)+i\epsilon_2(u)=-\frac{e^{-iu(b-ia)}}{2(K+1)}\left[K P+(1+2au)\bar{P}\right],$$

$$\sigma_1(u)+i\sigma_2(u)=\frac{i\,e^{-uc_0}e^{-iu(b+ia)}}{2(K+1)}\left[K\bar{P}-\{1+2u(c_0-a)\}P\right] \tag{151}$$

$$\tau_1(u)+i\tau_2(u)=\frac{e^{-uc_0}e^{-iu(b+ia)}}{2(K+1)}\left[K\bar{P}+\{1-2u(c_0-a)\}P\right]$$

These values enable us to find $I(z), J(z), H(z), K(z)$ from equations (146). We are interested in finding $\phi_a'(z)$. It may be directly seen from (147) that

$$\phi_a'(z)=\phi_0'(z)+\phi_1'(z)+\phi_2'(z)+\phi_3'(z)$$

$$= I(z)+J(z)+H(z)+K(z)$$

whence $\phi_a'(z)$ may be written as

$$\phi_a'(z)=\frac{1}{2\pi}\int_0^\infty\Big[e^{izu}\Big\{\alpha_1+\epsilon_2-\frac{\beta_2}{2}+\frac{\gamma_1}{2}+i\Big(\alpha_2-\epsilon_1+\frac{\beta_1}{2}+\frac{\gamma_2}{2}\Big)\Big\}$$
$$+e^{-izu}\Big\{\frac{\beta_2+i\beta_1}{2}+\frac{\gamma_1-i\gamma_2}{2}\Big\}\Big]du$$

Substituting for β_1, β_2, γ_1, γ_2 from (144), we get

$$\phi_a'(z)=\frac{1}{2\pi}\int_0^\infty\Big[\frac{e^{izu}}{s^2-t^2}\Big\{(s^2+cs+t)(\alpha_1+i\alpha_2)-(tc+ts+s)(\sigma_1+i\sigma_2)$$
$$+i\big[(s^2+cs-t)(\epsilon_1+i\epsilon_2)+(tc+ts-s)(\tau_1+i\tau_2)\big]\Big\}$$
$$+\frac{e^{-izu}}{s^2-t^2}\Big\{(s^2-cs-t)(\alpha_1-i\alpha_2)+(tc-ts+s)(\sigma_1-i\sigma_2)$$
$$-i\big[(s^2-cs+t)(\epsilon_1-i\epsilon_2)-(tc-ts-s)(\tau_1-i\tau_2)\big]\Big\}\Big]du$$

Substituting above from (161) for $\alpha_1+i\alpha_2$, $\epsilon_1+i\epsilon_2$, $\sigma_1+i\sigma_2$ and $\tau_1+i\tau_2$, we get after some simplification

$$\phi_a'(z)=\frac{i}{4\pi(K+1)}\int_0^\infty\Big[\frac{e^{izu-ibu}}{s^2-t^2}\Big\{P\big[Ke^{-au}(s^2+cs)+2u(c_0-a)e^{-u(c_0-a)}(ct+st)$$
$$+se^{-u(c_0-a)}\big]-\bar{P}\big[2au\,e^{-au}(s^2+ct)+te^{-au}+Ke^{-u(c_0-a)}(ct+st)\big]\Big\}$$
$$+\frac{e^{-izu+ibu}}{s^2-t^2}\Big\{-P\big[2aut\,e^{-au}+Kse^{-u(c_0-a)}+(sc-s^2)e^{-au}\big]$$
$$+\bar{P}\big[Kte^{-au}+(tc-ts)e^{-u(c_0-a)}+2u(c_0-a)se^{-u(c_0-a)}\big]\Big\}\Big]du \qquad (162)$$

Next we find $\psi_a'(z)$. This is obtained as follows : From (146) and (147)

$$\psi_a(z) = -z\,\phi_a'(z) + \phi_0(z) + \phi_2(z) - \phi_1(z) - \phi_3(z)$$

Now $\phi_a'(z)$ is known from (163), and $\phi_0(z), \phi_2(z), \phi_1(z), \phi_3(z)$ may be found by evaluating (146), whence $\psi_a(z)$ can be found. Applying these complex potential functions for $\phi_a'(z)$ and $\psi_a'(z)$ the stresses in the infinite strip are found by using the formula (11a) and (11b) i.e.

$$p_{xx} + p_{yy} = 4\,\mathrm{Re}\{\phi_a'(z)\}$$

$$p_{yy} - p_{xx} + 2i\,p_{xy} = 2\{\bar{z}\,\phi_a''(z) + \psi_a'(z)\}$$

whence p_{xx}, p_{xy}, p_{yy} are found. These are now superposed on the existing stress-field given in (149). Hence the problem of a strip under the action of a point force, and free from external tractions at its straight edges, is solved.

Next we consider the case of an infinite elastic strip under the action of a point force, when the straight boundaries are free from displacements.

To do this we solve an auxiliary problem.

Let the displacement on the boundary be prescribed as follows :

$$u_x^0 + i u_y^0 = f(x) + i F(x)$$

$$u_x^1 + i u_y^1 = g(x) + i G(x) \tag{153}$$

Here we use the same symbols $f(x), F(x);\ g(x), G(x)$ as in previous case since the treatment is similar. We again define $\alpha_1(u) + i\alpha_2(u),\ \epsilon_1(u) + i\epsilon_2(u);\ \sigma_1(u) + i\sigma_2(u),\ \tau_1(u) + i\tau_2(u)$ as in (143). The corresponding values of $\beta_1, \beta_2, \gamma_1, \gamma_2$ are given by

$$\beta_1 = \frac{1}{k^2s^2 - t^2}\Big[(tc-ks)\left[\bar{e}^{t}\{(t-k)\alpha_2 + t\epsilon_1\} + k\sigma_2\right] + st\left[\bar{e}^{t}\{(t+k)\epsilon_1 + t\alpha_2\} - k\tau_1\right]\Big]$$

$$\beta_2 = \frac{1}{k^2s^2 - t^2}\Big[(tc-ks)\left[\bar{e}^{t}\{(k-t)\alpha_1 + t\epsilon_2\} - k\sigma_1\right] + st\left[e^{-t}\{-(k+t)\epsilon_2 + t\alpha_1\} + k\tau_2\right]\Big]$$

$$\gamma_1 = \frac{1}{k^2s^2 - t^2}\Big[(tc+ks)\left[\bar{e}^{t}\{-(k+t)\epsilon_2 + t\alpha_1\} + k\tau_2\right] - ts\left[\bar{e}^{t}\{(k-t)\alpha_1 + t\epsilon_2\} - k\sigma_1\right]\Big] \tag{154}$$

$$\gamma_2 = \frac{1}{k^2s^2 - t^2}\Big[(tc+ks)\left[\bar{e}^{t}\{(k+t)\epsilon_1 + t\alpha_2\} - k\tau_1\right] - ts\left[\bar{e}^{t}\{(t-k)\alpha_2 + t\epsilon_1\} + k\sigma_2\right]\Big]$$

and symbols $I(z)$, $J(z)$, $H(z)$, $K(z)$ have the same fundamental value as has been defined in the previous case by equation (145).

In this case the functions $\phi_r(z)$, $(r=0,1,2,3)$ and $\psi_r(z)$, $(r=0,1,2,3)$ are evaluated as follows :

$$\phi_0(z) = \frac{\mu}{\pi k}\int_0^\infty e^{izu}\{\alpha_1(u)+i\alpha_2(u)\}\,du \; , \quad \psi_0(z) = -z\phi_0'(z) - k\phi_0(z)$$

$$\phi_1(z) = \frac{\mu i}{\pi k}\int_0^\infty e^{izu}\{\epsilon_1(u)+i\epsilon_2(u)\}\,du \; , \quad \psi_1(z) = -z\phi_1'(z) + k\phi_1(z)$$

$$\phi_2(z) = \frac{2\mu}{k} H(z) \qquad \psi_2(z) = -z\phi_2'(z) - k\phi_2(z) \tag{155}$$

$$\phi_3(z) = \frac{2\mu}{k} K(z) \qquad \psi_3(z) = -z\phi_3'(z) + k\phi_3(z)$$

whence we obtain $\phi_a(z)$ and $\psi_a(z)$ by the same formulas given in equations (147).

As in previous case to begin with, we imagine that this point-force is acting in an infinite medium. This gives rise to certain displacement everywhere which is given by

$$4\pi\mu(k+1)u_x = \frac{-2X(y-a)^2+2Y(x-b)(y-a)}{(x-b)^2+(y-a)^2} + (1-k)X - kX\log\left[(x-b)^2+(y-a)^2\right]$$

$$4\pi\mu(k+1)u_y = \frac{2Y(y-a)^2+2X(x-b)(y-b)}{(x-b)^2+(y-a)^2} - (1+k)Y - kY\log\left[(x-b)^2+(y-a)^2\right] \tag{156}$$

If the boundary is to be displacement free, we must nullify the displacements $u_x^0, u_y^0, u_x^1, u_y^1$ given by (156) by putting $y=0$ and $y=c_0$.

It is seen from (156) that the displacements to be nullified contain terms which are infinite at infinity. However, Tiffen has shown ((32)) that the potentials,

$$\phi(z) = \frac{P}{2\pi(K+1)} \log(z-b+ia)$$

$$\psi(z) = -2\phi'(z) + \frac{P-K\bar{P}}{2\pi(K+1)} + \frac{-K\bar{P}}{2\pi(K+1)} \log(z-b+ia)$$

called 'image potentials' remove infinite terms in the displacement along $y=0$ and contribute zero displacement along $y=0$. These potentials also remove the non-evanescent displacement along $y=c_0$. These give rise to the following displacements :

$$4\pi\mu(K+1)u_x = \frac{2y[X(y+a)-Y(x-b)]}{(x-b)^2+(y+a)^2} - (1-K)X + KX\log[(x-b)^2+(y+a)^2]$$

(157)

$$4\pi\mu(K+1)u_y = \frac{-2y[X(x-b)+Y(y+a)]}{(x-b)^2+(y+a)^2} + (1+K)Y + KY\log[(x-b)^2+(y+a)^2]$$

We superpose displacement (157) over that in (156) to get the displacements to be nullified. Thus to free the straight edges of the strip from displacements,

we require the complex potentials which satisfy the boundary conditions (given by taking equal and opposite displacements to that given by (156) and (157) on $y=0$ and $y=c_0$).

$$2\pi(k+1)u_x^0 = \frac{aY(x-b)+a^2X}{(x-b)^2+a^2},$$

$$2\pi(k+1)u_y^0 = \frac{aX(x-b)-a^2Y}{(x-b)^2+a^2},$$

$$2\pi(k+1)u_x^1 = \frac{X(c_0-a)^2-Y(c_0-a)(x-b)}{(x-b)^2+(c_0-a)^2} + \frac{-c_0X(c_0+a)+c_0Y(x-b)}{(x-b)^2+(c_0+a)^2} + \frac{kX}{2}\log\frac{(x-b)^2+(c_0+a)^2}{(x-b)^2+(c_0-a)^2}, \qquad (158)$$

$$2\pi(k+1)u_y^1 = \frac{-X(c_0-a)(x-b)-Y(c_0-a)^2}{(x-b)^2+(c_0-a)^2} + \frac{c_0X(x-b)+c_0Y(c_0+a)}{(x-b)^2+(c_0+a)^2} + \frac{kY}{2}\log\frac{(x-b)^2+(c_0+a)^2}{(x-b)^2+(c_0-a)^2}$$

Hence we solve the problem of an infinite strip with the boundary displacements given by equations (158). These give us the values of $f(x), F(x)$; $g(x)$, $G(x)$ from the equations (153) whence $\alpha_1(u)+i\alpha_2(u)$, $\epsilon_1(u)+i\epsilon_2(u)$, $\sigma_1(u)+i\sigma_2(u)$ $\tau_1(u)+i\tau_2(u)$ are evaluated using equation (143). The results are as follows :

$$\alpha_1 + i\alpha_2 = \frac{a}{2\mu(K+1)} \; \bar{P} \, e^{-iu(b-ia)}$$

$$\epsilon_1 + i\epsilon_2 = \frac{-ia}{2\mu(K+1)} \; \bar{P} \, e^{-iu(b-ia)}$$

$$\sigma_1 + i\sigma_2 = \frac{e^{-t}}{2\mu(K+1)} \left[P e^{-ibu} (2c_0 \sinh au - a e^{au}) - \frac{2K}{u} \sinh au \, Y e^{-ibu} \right] \qquad (159)$$

$$\tau_1 + i\tau_2 = \frac{i e^{-t}}{2\mu(K+1)} \left[P e^{-ibu} (2c_0 \sinh au - a e^{au}) + \frac{2K}{u} \, i \, Y \sinh au \, e^{ibu} \right]$$

By the process indicated earlier we obtain $\phi_a(z)$ and $\psi_a(z)$. The function $\phi_a(z)$ is obtained as follows :

It is derived from (155) that additional complex potential function, distinguished by subscript a, is given by

$$\phi_a(z) = \frac{\mu}{K\pi} \int_0^\infty \Big[e^{izu} \Big\{ (\alpha_1 + i\alpha_2) + \frac{\sigma_1 + i\sigma_2}{2} + \frac{i(\beta_1 + i\beta_2)}{2} + i(\epsilon_1 + i\epsilon_2) \Big\}$$
$$+ e^{-izu} \Big\{ \frac{i(\beta_1 - i\beta_2)}{2} + \frac{\sigma_1 - i\sigma_2}{2} \Big\} \Big] du$$

Substituting in this expression from (154) for $\beta_1, \beta_2 ; \ \sigma_1 , \sigma_2$, we have

$$\phi_a(z) = \frac{\mu}{2K\pi} \int_0^\infty \frac{e^{izu}}{(K^2 s^2 - t^2)} \Big[(\alpha_1 + i\alpha_2) \{ K^2 (c+s) - Kt \} + i(\epsilon_1 + i\epsilon_2) \{ K^2 s (c+s) + Kt \}$$
$$+ (\sigma_1 + i\sigma_2) \{ Kt(c+s) - K^2 s \} - i(\tau_1 + i\tau_2) \{ Kt(c+s) + K^2 s \} \Big] +$$

$$+\frac{e^{-izu}}{k^2s^2-t^2}\Big[(\alpha_1-i\alpha_2)\left\{-k^2s(c-s)+kt\right\}-i(\epsilon_1-i\epsilon_2)\left\{k^2s(c-s)+kt\right\}$$

$$-(\sigma_1-i\sigma_2)\left\{kt(c+s)-k^2s\right\}+i(\tau_1-i\tau_2)\left\{kt(c-s)+k^2s\right\}\Big]du \qquad (160)$$

Now substituting (158) into (160) for $\alpha_1+i\alpha_2$, $\epsilon_1+i\epsilon_2$, $\sigma_1+i\sigma_2$, $\tau_1+i\tau_2$ we get

$$\phi_a(z)=\frac{1}{\pi k(k+1)}\int_0^\infty \frac{e^{izu}}{k^2s^2-t^2}\Big[2ak^2s(c+s)\bar{P}\,e^{-iu(b-ia)}$$

$$+2ktP\left\{c_0e^{-iu(b+ia)}-c_0e^{-iu(b-ia)}-ae^{-iu(b+ia)}\right\}$$

$$-k^2c_0\bar{P}\left\{e^{-iu(b+ia)}-e^{-iu(b-ia)}\right\}+\frac{k^3s(c-s)}{u}P\left\{e^{-iu(b+ia)}-e^{-iu(b-ia)}\right\}$$

$$+\frac{e^{-izu}}{k^2s^2-t^2}\Big[2aktPe^{iu(b+ia)}+2k^2s(c-s)\bar{P}\left\{c_0e^{iu(b-ia)}-c_0e^{iu(b+ia)}-ae^{iu(b-ia)}\right\}$$

$$+k^2c_0(c-s)^2\bar{P}\left\{e^{iu(b-ia)}-e^{iu(b+ia)}\right\} \qquad (161)$$

$$-\frac{k^3s(c-s)}{u}P\left\{e^{iu(b-ia)}-e^{iu(b+ia)}\right\}\Big]\,du$$

The function $\psi_a(z)$ is obtained from (155)

$$\psi_a(z) = \psi_0(z) + \psi_1(z) + \psi_2(z) + \psi_3(z)$$

$$= -z\,\phi_a'(z) + k\left\{\phi_1(z) + \phi_3(z) - \phi_0(z) - \phi_2(z)\right\}$$

Since, $\phi_a'(z)$ is known from (161) and $\phi_0(z), \phi_1(z), \phi_2(z), \phi_3(z)$ are given by (155), the function $\psi_a(z)$ can be easily found.

We now evaluate the stress-field given by

$$p_{xx} + p_{yy} = 4\,\mathrm{Re}\left\{\phi_a'(z)\right\}$$

$$p_{yy} - p_{xx} + 2i\,p_{xy} = 2\left[\bar{z}\,\phi_a''(z) + \psi_a'(z)\right]$$

and superpose on already existing stress (149), thereby obtaining complete stress field due to a point force in the strip when its boundary is free from displacements.

CHAPTER X

CIRCULAR INCLUSION IN AN INFINITE ELASTIC STRIP-I

We consider the following problem :

In an infinite elastic strip a symmetrically situated circular inclusion tends to undergo a spontaneous deformation. Due to the presence of outside region (called matrix) the stresses develop both in the inclusion and the matrix. The problem is to find the stress and displacement fields.

This work may be compared with earlier solutions relating to inclusion problems, namely ((9, 10, 13, 14)), where the region has been infinite, and of Bhargava and Kapoor ((15)) and Bhargava and Sharma (unpublished work reported in this thesis) dealt with the problem of the inclusion in half-plane. In this chapter we consider (for the first time to the knowledge of the author), the problem of a circular inclusion in an infinite strip.

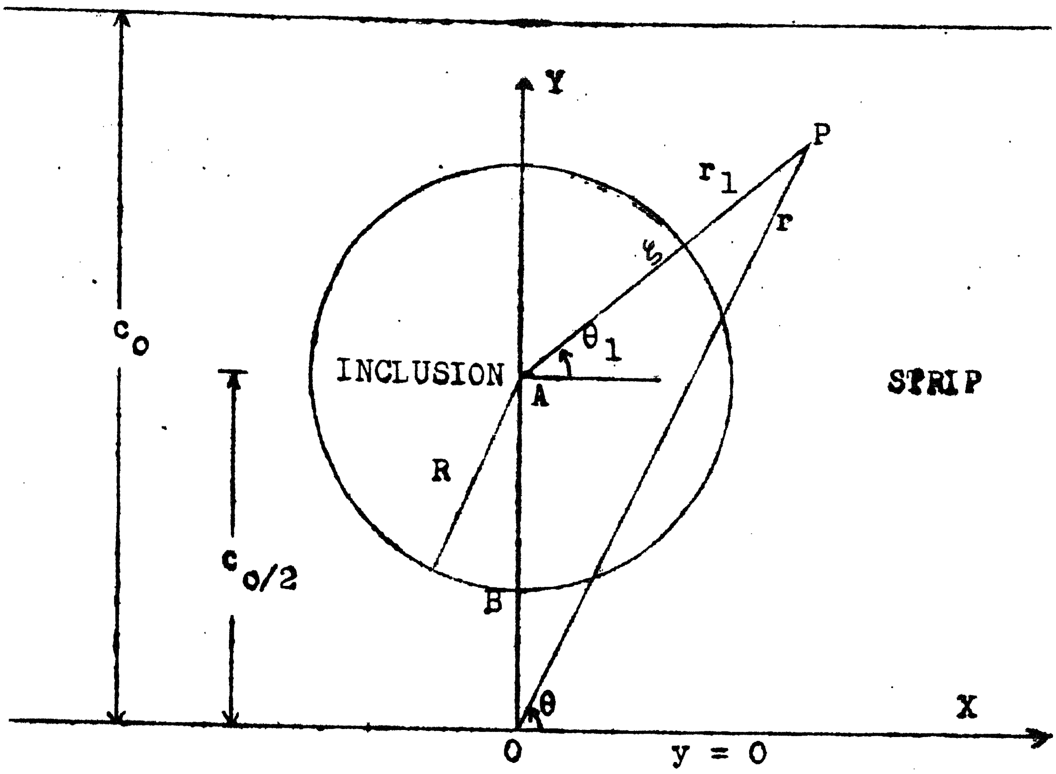

Figure 3, Circular Inclusion in infinite elastic strip coordinate . system.

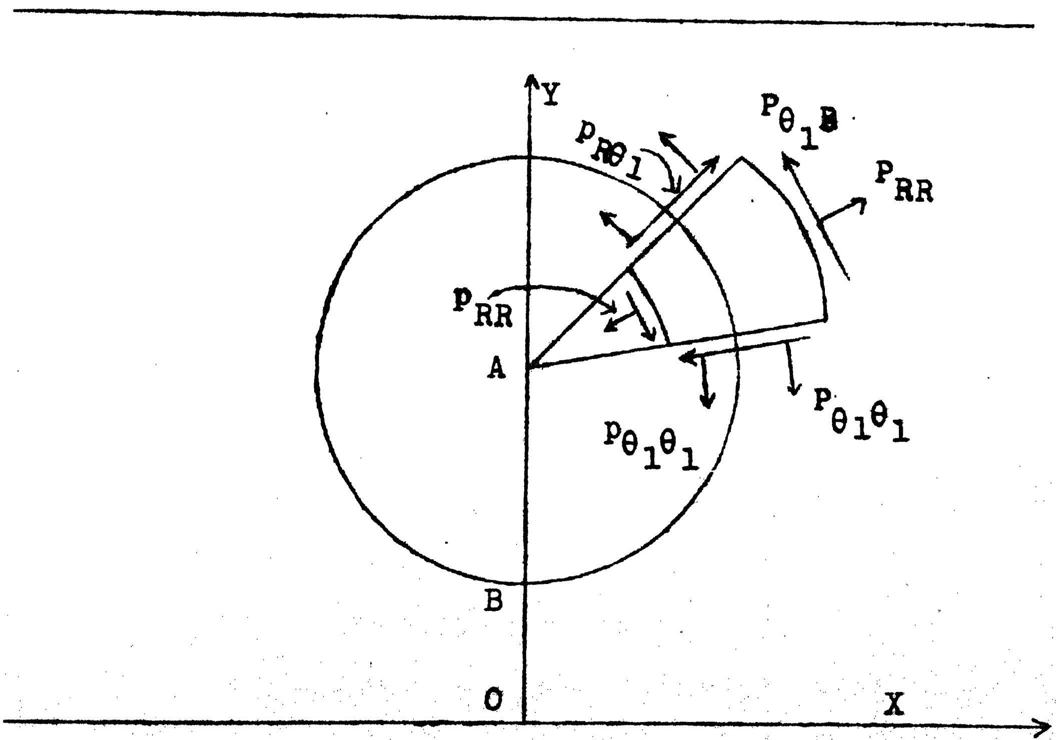

Figure 4, A schematic view of normal and shear stress distribution in inclusion and strip.

Choosing z and ξ reference systems in accordance with the figure 3 page 118, it is seen that the boundary of the circle of radius R is given by $\xi\bar{\xi}=R^2$. The centre of the circle is symmetrically situated within the strip. One edge of the strip has been taken as real axis and a perpendicular line to it in the plane of the strip passing through the centre of the circle has been taken as imaginary axis. The strip is bound by the lines $y=0$ and $y=c_0$ and extends from $-\infty$ to $+\infty$ in x-direction.

The inclusion in absence of the sorrounding strip tends to undergo uniform prescribed deformation

$$e_{xx}=\delta \ , \quad e_{yy}=\delta \ , \quad e_{xy}=0$$

where δ lies within the limits of the classical theory of elasticity. The 'free inclusion' state is not achieved due to the constraints of the strip. Thus locked up-accomodation stresses arise both in the inclusion and the matrix. The stress and displacement fields inside the inclusion and the strip have been evaluated in this paper. Results are given in the tables given in the appendix, following this chapter.

The problem has been solved by utilising the concept explained in chapter 11 page 14. The procedure is as follows :

First we solve the problem when the circular inclusion is present in the infinite medium. It has its centre at $z = ic_0/2$ and radius R. At this stage if we consider the strip $0 \leq y \leq c_0$ and $-\infty < x < +\infty$ in the infinite region, the tractions would be present at the edges $y = 0$ and $y = c_0$. These tractions we nullify by superposing, equal and opposite tractions, thus obtaining the solution of a circular inclusion in an elastic strip with the traction-free edges.

The solution of the problem in the infinite region is well-known. The solution is given in chapter II, page 18,

$$\phi_i'(z) = \frac{2(\lambda+\mu)}{K+1}\delta \quad , \quad \psi_i'(z) = 0 \; ,$$

$$\phi_m'(z) = 0 \quad , \quad \psi_m'(z) = \frac{2(K-1)(\lambda+\mu)\delta}{K+1}\,\frac{R^2}{z_1^2} \tag{162}$$

where $z_1 = z - ic_0/2$.

It can be shown that the normal and shearing stresses p_{RR}, $p_{R\theta_1}$ at the equilibrium interface at this stage are

$$p_{RR}^b - i\,p_{R\theta_1}^b = P_{RR}^b - i\,P_{R\theta_1}^b = -\frac{2(\lambda+\mu)(K-1)\delta}{K+1} , \tag{163}$$

where we have used the obvious notations p^b_{RR}, $p^b_{R\theta_1}$ for normal and shearing stresses on the inclusion boundary $|\xi| = R$. Also on the boundary the hoop stress in the inclusion and the matrix are given by

$$p^b_{\theta_1\theta_1} = -\frac{2(\lambda+\mu)\,\delta(k-1)}{k+1},$$

$$P^b_{\theta_1\theta_1} = \frac{2(\lambda+\mu)\,\delta\,(k-1)}{k+1}. \tag{164}$$

As indicated in the previous chapter, the effect of a point force $P = X+iY$ acting at $b+ia$, giving traction free boundary or the leading edges, can be expressed in terms of complex potential (162), because the other function $\psi_a(z)$ is related to $\phi_a(z)$ by the following relation

$$\psi_a(z) = -z\,\phi_a'(z) + \{\phi_0(z) + \phi_2(z) - \phi_1(z) - \phi_3(z)\} \tag{165}$$

where $\phi_r(z)$, $\{r = 0, 1, 2, 3\}$ are given in (146).

The cumulative effect of continuous distribution of layer of point forces acting along Γ in the strip, instead of a single point force may be obtained by integrating the expression (162) along the contour Γ. After some calculations, we arrive at the simple expression

$$\phi'(z) = \frac{(\kappa-1)(\lambda+\mu)\delta R^2}{\kappa+1}\int_0^\infty \frac{u}{s+t}\left[e^{izu+uc_0/2} + e^{-izu-uc_0/2}\right]du \qquad (166)$$

Differentiating (165), we get

$$\psi_a'(z) = -z\,\phi_a''(z) - 2\left\{\phi_1'(z) + \phi_3'(z)\right\} \qquad (167)$$

To find $\psi'(z)$ when the layer of point force is present, we substitute the values of $\phi'(z)$ and $\phi_1'(z)$, $\phi_3'(z)$ from (145) in (167) and integrate round the contour Γ. After some calculation we obtain

$$\psi'(z) = -\frac{(\kappa-1)(\lambda+\mu)\delta R^2}{\kappa+1}\int_0^\infty \frac{u}{s+t}\Big[e^{izu-uc_0/2}(s+c+1) + e^{-izu-uc_0/2}(c-s-2t+1)$$

$$+izu\left\{e^{izu+uc_0/2} - e^{-izu-uc_0/2}\right\}\Big]du \qquad (168)$$

To get the stress-field in the inclusion, we add the following three stress fields (i) $p_{xx} = -2(\lambda+\mu)\delta$, $p_{xy} = 0$ $p_{yy} = -2(\lambda+\mu)\delta$ which is obtained by reducing it to the size of hole (ii) stress-field due to infinite matrix which shall be obtained by using (162) and the equations (11a,b) (iii) additional stress field given by complex potentials (166) and (168) due to the infinite strip,

because its leading edges are to be stress-free.

The stress-field for the remaining part of the strip may be obtained by superposing the stresses due to complex potentials $\phi'_m(z)$, $\psi'_m(z)$ in (162) and (166), (168).

Explicitly speaking the stresses which generate everywhere from (166), (168), are the following

$$p_{xx} + p_{yy} = P_{xx} + P_{yy} = 4\,\mathrm{Re}\{\phi'(z)\}$$

$$= \frac{8(k-1)(\lambda+\mu)\delta R^2}{k+1}\int_0^\infty \frac{u\cos(ur\cos\theta)\cosh u(r\sin\theta - c_0/2)}{s+t}\,du \tag{169}$$

$$p_{yy} - p_{xx} + 2i\,p_{xy} = P_{yy} - P_{xx} + 2i\,P_{xy} = 2\{\bar{z}\,\phi''(z) + \psi'(z)\}$$

$$= -\frac{4(\lambda+\mu)(k-1)\delta R^2}{k+1}\int_0^\infty \frac{u}{s+t}\Big[(1 - uc_0 + e^{-uc_0})\{\cos(ur\cos\theta)\times$$

$$\times\cosh u(r\sin\theta - c_0/2) - i\sin(ur\cos\theta)\sinh u(r\sin\theta - c_0/2)\} \tag{170}$$

$$+ 2u(r\sin\theta - c_0/2)\{\cos(ur\cos\theta)\sinh u(r\sin\theta - c_0/2)$$

$$- i\sin(ur\cos\theta)\cosh u(r\sin\theta - c_0/2)\}\Big]\,du$$

Since the additional stress-field is same for both the matrix and the inclusion and so also are the elastic

constants of the materials, there is perfect bond on the common interface. The Cartesian stress components given by these additional complex potentials (166), (168), are

$$\frac{(K+1)p_{xx}}{2(K-1)(\lambda+\mu)\delta} = R^2\int_0^\infty \frac{u\cos(ur\cos\theta)}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 + 3)\cosh u(r\sin\theta - c_0/2)$$

$$+ 2u(r\sin\theta - c_0/2)\sinh u(r\sin\theta - c_0/2)\Big]du$$

$$\frac{(K+1)p_{yy}}{2(K-1)(\lambda+\mu)\delta} = R^2\int_0^\infty \frac{u\cos(ur\cos\theta)}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 - 1)\cosh u(r\sin\theta - c_0/2)$$

$$+ 2u(r\sin\theta - c_0/2)\sinh u(r\sin\theta - c_0/2)\Big]du$$

$$\frac{(K+1)p_{xy}}{2(K-1)(\lambda+\mu)\delta} = R^2\int_0^\infty \frac{u\sin(ur\cos\theta)}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 + 1)\sinh u(r\sin\theta - c_0/2)$$

$$+ 2u(r\sin\theta - c_0/2)\cosh u(r\sin\theta - c_0/2)\Big]du \qquad (171)$$

These give the additional stress field to that given by the complex potentials (162).

Using relations (12), the resultant normal, tangential and hoop stresses have been evaluated. The values of these stress components are given by

$$p^b_{RR} = P^b_{RR} = \frac{2(K-1)(\lambda+\mu)\delta R^2}{K+1}\int_0^\infty \frac{u}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 + 1)\times$$

$$\times\left\{\cos(Ru\cos\theta_1)\cos 2\theta_1 \cosh(Ru\sin\theta_1) + \sin 2\theta_1 \sin(Ru\cos\theta_1)\sinh(Ru\sin\theta_1)\right\}$$

$$+ 2u\sin\theta_1\left\{\cos 2\theta_1 \cos(Ru\cos\theta_1)\sinh(Ru\sin\theta_1) + \sin 2\theta_1 \sin(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\right\}$$

$$+ 2\cos(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\Big]du - \frac{2(K-1)(\lambda+\mu)\delta}{K+1}$$

$$p^b_{R\theta_1} = P^b_{R\theta_1} = -\frac{2(K-1)(\lambda+\mu)\delta R^2}{K+1}\int_0^\infty \frac{u}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 + 1)\times$$

$$\times\left\{\sin 2\theta_1 \cos(Ru\cos\theta_1)\cosh(Ru\sin\theta_1) - \cos 2\theta_1 \sin(Ru\cos\theta_1)\sinh(Ru\sin\theta_1)\right\} \tag{172}$$

$$+ 2u\sin\theta_1\left\{\sin 2\theta_1 \cos(Ru\cos\theta_1)\sinh(Ru\sin\theta_1) - \cos 2\theta_1 \sin(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\right\}\Big]du$$

$$p^b_{\theta_1\theta_1} = P^b_{\theta_1\theta_1} - \frac{4(K-1)(\lambda+\mu)\delta}{K+1}$$

$$= -\frac{2(K-1)(\lambda+\mu)\delta R^2}{K+1}\int_0^\infty \frac{u}{\sinh uc_0 + uc_0}\Big[(e^{-uc_0} - uc_0 + 1)\left\{\cos 2\theta_1 \cos(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\right.$$

$$\left. + \sin 2\theta_1 \sin(Ru\cos\theta_1)\sinh(Ru\sin\theta_1)\right\}$$

$$+ 2u\sin\theta_1\left\{\cos 2\theta_1 \cos(Ru\cos\theta_1)\sinh(Ru\sin\theta_1) + \sin 2\theta_1 \sin(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\right\}$$

$$- 2\cos(Ru\cos\theta_1)\cosh(Ru\sin\theta_1)\Big]du - \frac{2(K-1)(\lambda+\mu)\delta}{K+1} \tag{173}$$

We give in the appendix the tables containing the values of the boundary stresses. First column gives the angle θ_1 ranging from 0 to 90° with an interval of 10°. The second and third columns give the corresponding values of normal and shearing stresses over the inclusion, which are the same as for matrix, because of their continuity property. The hoop stresses have separately been tabulated. First column gives θ_1 as above. The second and third columns give hoop stresses inside and outside respectively. The Poisson's ratio has been taken to be equal to 1/3.

The values of $c_o = L$ have been taken to be equal to 3, 4, 5, 6, 7, 8, 9, 10,... which in effect means that the leading edges are at a distance of 3/2, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5... times the radius of the inclusion.

In the next chapter the case of a deforming inclusion in an infinite strip is considered, but the strip in this case, is constrained so that the displacement is zero along the straight edges.

Appendix to Chapter X

TABLE 1

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=3	
0	-2.31491229	-0.00000000
10	-2.31977596	-0.04246913
20	-2.32513708	-0.06527332
30	-2.31675568	-0.06795858
40	-2.25567265	-0.03671937
50	-2.15680179	0.01573502
60	-1.97636294	0.06266207
70	-1.76968490	0.07830304
80	-1.58066487	0.05251081
90	-1.50905317	-0.00000002
	L=6	
0	-2.69811967	-0.00000000
10	-2.69663998	0.00599790
20	-2.68957941	0.01834628
30	-2.66984209	0.04104321
40	-2.62848878	0.07221345
50	-2.55956998	0.10317252
60	-2.46644795	0.11998761
70	-2.36623570	0.10973587
80	-2.28722620	0.06533014
90	-2.25702645	-0.00000001
	L=9	
0	-2.86773968	-0.00000000
10	-2.86458089	0.01644796
20	-2.85515153	0.03497626
30	-2.83563901	0.05601195
40	-2.80326009	0.07710020
50	-2.75725451	0.09270040
60	-2.70164520	0.09562427
70	-2.64627767	0.08021692
80	-2.60476016	0.04608914
90	-2.58928940	-0.00000000

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=5	
0	-2.95597911	-0.00000000
10	-2.95306134	0.01708189
20	-2.94382092	0.03440716
30	-2.92713044	0.05131265
40	-2.90215585	0.06566108
50	-2.86962783	0.07395599
60	-2.83291870	0.07234174
70	-2.79819356	0.05827457
80	-2.77390811	0.03275651
90	-2.76377985	-0.00000000
	L=7	
0	-3.00746551	-0.00000000
10	-3.00484583	0.01530193
20	-2.99680629	0.03012591
30	-2.98300833	0.04350013
40	-2.96349973	0.05370109
50	-2.93940005	0.05842738
60	-2.91398772	0.05547718
70	-2.88960460	0.04376294
80	-2.87270372	0.02421278
90	-2.86686271	-0.00000000
	L=9	
0	-3.04008773	-0.00000000
10	-3.03782403	0.01316343
20	-3.03099695	0.02559122
30	-3.01962208	0.03628024
40	-3.00409541	0.04385142
50	-2.98556486	0.04671659
60	-2.96616113	0.04353869
70	-2.94884124	0.03385149
80	-2.93677798	0.01855854
90	-2.93244442	-0.00000000

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=9	
0	-3.06206068	-0.00000000
10	-3.06012410	0.01121512
20	-3.05434686	0.02163194
30	-3.04490408	0.03031099
40	-3.03231019	0.03613743
50	-3.01763424	0.03798677
60	-3.00259395	0.03494397
70	-2.98939911	0.02690293
80	-2.98331977	0.01465696
90	-2.97707832	-0.00000000
	L=10	
0	-3.07797074	-0.00000000
10	-3.07591292	0.00956857
20	-3.07100347	0.01830764
30	-3.06308359	0.02551849
40	-3.05269232	0.02913714
50	-3.04078847	0.02135613
60	-3.02877933	0.02560580
70	-3.01837942	0.02186952
80	-3.01128778	0.01186143
90	-3.00876778	-0.00000000
	L=11	
0	-3.08893099	-0.00000000
10	-3.08750474	0.00820971
20	-3.08390519	0.01569067
30	-3.07658878	0.02168607
40	-3.06788991	0.02543707
50	-3.05803770	0.02627921
60	-3.04822135	0.02381937
70	-3.03980386	0.01811642
80	-3.03410387	0.00979331
90	-3.03208569	-0.00000000

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=12	
0	-3.09750378	-0.00000000
10	-3.09626868	0.00709405
20	-3.09264427	0.01352013
30	-3.08689278	0.01860721
40	-3.07950315	0.02171361
50	-3.07122511	0.02231334
60	-3.06304744	0.02012593
70	-3.05608833	0.01524742
80	-3.05140162	0.00822162
90	-3.04974690	-0.00000000
	L=13	
0	-3.10413420	-0.00000000
10	-3.10305712	0.00617576
20	-3.09990567	0.01174463
30	-3.09493157	0.01611101
40	-3.08858508	0.01872683
50	-3.08152905	0.01916513
60	-3.07460893	0.01722091
70	-3.06879485	0.01302692
80	-3.06482971	0.00699974
90	-3.06344703	-0.00000000
	L=14	
0	-3.10936913	-0.00000000
10	-3.10842329	0.00541549
20	-3.10566223	0.01028145
30	-3.10132271	0.01406779
40	-3.09581620	0.01630136
50	-3.08973068	0.01662892
60	-3.08379629	0.01489726
70	-3.07880113	0.01122476
80	-3.07546341	0.00603124
90	-3.07428983	-0.00000000

TABLE 2

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=3	
0	-2.43568916	3.82749611
10	-2.43212974	3.85105593
20	-2.36436740	3.91881738
30	-2.26117676	4.02256848
40	-2.13546804	4.14674193
50	-2.00601834	4.27716690
60	-1.88274227	4.40144302
70	-1.77544977	4.50773549
80	-1.69667727	4.58650804
90	-1.66659359	4.61659179
	L=4	
0	-2.59559223	3.68759304
10	-2.59786631	3.69531597
20	-2.58703013	3.71615517
30	-2.55965849	3.74352688
40	-2.51497981	3.76820546
50	-2.50182092	3.78196435
60	-2.50447860	3.77870675
70	-2.51952165	3.76366362
80	-2.53651573	3.74666855
90	-2.54386848	3.73931679
	L=5	
0	-2.73561612	3.54756916
10	-2.73399305	3.54919222
20	-2.73037907	3.55280620
30	-2.72811687	3.55506840
40	-2.73136774	3.55181754
50	-2.74313119	3.54005408
60	-2.76312909	3.52005619
70	-2.78671008	3.49647519
80	-2.80600870	3.47717661
90	-2.81348282	3.46970249

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=5	
0	-2.83639988	3.44678539
10	-2.83694875	3.44623652
20	-2.83920610	3.44397917
30	-2.84471214	3.43847314
40	-2.85511383	3.42807144
50	-2.87101984	3.41216543
60	-2.89097939	3.39221180
70	-2.91116902	3.37201625
80	-2.92639983	3.35678548
90	-2.93208665	3.35109863
	L=7	
0	-2.90654072	3.37664455
10	-2.90780997	3.37537530
20	-2.91189271	3.37129256
30	-2.91942626	3.36375901
40	-2.93090066	3.35228461
50	-2.94600293	3.33718234
60	-2.96311381	3.32007146
70	-2.97930133	3.30388394
80	-2.99101257	3.29217270
90	-2.99529925	3.28788602
	L=8	
0	-2.95608172	3.32710396
10	-2.95752543	3.32565984
20	-2.96195960	3.32122570
30	-2.96957368	3.31361160
40	-2.98032883	3.30285648
50	-2.99357805	3.28960723
60	-3.00781968	3.27536559
70	-3.02078220	3.26240307
80	-3.02992639	3.25325888
90	-3.03323203	3.24995324

θ	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=9	
0	-2.99194777	3.29128759
10	-2.99335998	3.28902529
20	-2.99761119	3.28557411
30	-3.00466880	3.27851690
40	-3.01425660	3.26892871
50	-3.02563971	3.25755237
60	-3.03747532	3.24570996
70	-3.04798979	3.23519549
80	-3.05528373	3.22790153
90	-3.05789828	3.22528699
	L=10	
0	-3.01857689	3.26460838
10	-3.01988462	3.26330066
20	-3.02377766	3.25940761
30	-3.03011471	3.25307056
40	-3.03852123	3.24468404
50	-3.04829941	3.23492587
60	-3.05818161	3.22500366
70	-3.06684202	3.21634328
80	-3.07277930	3.21040598
90	-3.07489495	3.20829149
	L=11	
0	-3.03881246	3.24437281
10	-3.03999659	3.24318868
20	-3.04349610	3.23968917
30	-3.04912028	3.23406500
40	-3.05646330	3.22672197
50	-3.06482993	3.21835533
60	-3.07322660	3.20995867
70	-3.08046558	3.20271969
80	-3.08538571	3.19779956
90	-3.08713108	3.19605419

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=12	
0	-3.05451101	3.22867426
10	-3.05557382	3.22761175
20	-3.05869[illegible]	3.22449727
30	-3.06367484	3.21951044
40	-3.07009959	3.21308568
50	-3.07733279	3.20585248
60	-3.08451134	3.19867393
70	-3.09064333	3.19254196
80	-3.09476398	3.1884[illegible]134
90	-3.09624782	3.18693745
	L=13	
0	-3.06691414	3.21627113
10	-3.06786498	3.21532079
20	-3.07065108	3.21253419
30	-3.07505995	3.20812532
40	-3.08070403	3.20248124
50	-3.08700141	3.19618386
60	-3.09319833	3.18998694
70	-3.09845436	3.18473092
80	-3.10198557	3.18119970
90	-3.10323074	3.17995453
	L=14	
0	-3.07687238	3.20631289
10	-3.07772380	3.20546147
20	-3.08021179	3.20297348
30	-3.08412948	3.19905579
40	-3.08911267	3.19407260
50	-3.09463412	3.18855116
60	-3.10003158	3.18315369
70	-3.10458410	3.17860118
80	-3.10769043	3.17555484
90	-3.10873239	3.17448288

CHAPTER XI

CIRCULAR INCLUSION IN AN INFINITE ELASTIC STRIP-II

Consider the problem of a circular inclusion in an infinite elastic strip. The straight boundary of the strip is free from displacements. The treatment is similar to the case of a strip free from tractions, discussed in the preceding chapter.

We first consider the case when the inclusion is present in an infinite medium, and calculate the displacements every-where and specially on the boundary of the strip. We impose equal and opposite displacements on the straight edges. This nullifies the displacements on the boundary and gives the solution for the case when the inclusion is present in an infinite strip and the edges of the strip are free from displacements.

Using the same notation and coordinate system as in the preceding chapter, it is easy to see that

the resulting complex potentials for the inclusion and infinite medium are given by (162).

Following the analysis given in chapter IX it is seen that the additional complex potentials are given, in this case, by equation (161). These, when superposed on the complex potentials given by (162) will give the effect of point-force in an infinite strip with the boundaries free from displacements.

When there is an inclusion, the cumulative effect of a layer of point-forces acting along Γ in the strip is to be evaluated. This is given by

$$\phi(z) = \int_{\Gamma} \phi_a(z)\, ds \tag{174}$$

where ds denotes arc differential along Γ and $\phi_a(z)$ is given by (161). It may be stated that in this chapter the additional complex potential due to a single point force is distinguished by subscript a, where as resultant additional complex potential, due to cumulative effect, has been denoted by the function $\phi(z)$.

After suppressing the details of integration along contour of integration Γ, we get

$$\phi(z) = \frac{i(\lambda+\mu)\delta}{\kappa+1}\int_0^\infty\left[\frac{e^{izu}}{\kappa^2s^2-t^2}\left\{-(\kappa s+t)e^{u\omega/2}+2t^2e^{-u\omega/2}+\kappa te^{u\omega/2}+\kappa^2se^{-3u\omega/2}\right\}\right.$$

$$\left.+\frac{e^{-izu}}{\kappa^2s^2-t^2}\left\{(\kappa s+t)e^{-u\omega/2}-2\kappa se^{-3u\omega/2}-t\kappa e^{-5u\omega/2}-\kappa^2se^{-u\omega/2}\right\}\right]du \qquad (175)$$

As is evident from analysis described in chapter IX, it is enough to find the complex potential $\phi'_a(z)$ only and the second one, $\psi'_a(z)$ is related to the latter as given in relations (166) for a single point force. The value of $\psi(z)$ due to continuous distribution of point-forces is given by

$$\psi(z) = \frac{(\lambda+\mu)\delta}{\kappa+1}\int_0^\infty\left[\frac{zue^{izu}}{\kappa^2s^2-t^2}\left\{-(\kappa s+t)e^{u\omega/2}+2t^2e^{-u\omega/2}+\kappa te^{u\omega/2}+\kappa^2se^{-3u\omega/2}\right\}\right.$$

$$\left.-\frac{zue^{-izu}}{\kappa^2s^2-t^2}\left\{(\kappa s+t)e^{-u\omega/2}-2\kappa se^{-3u\omega/2}-t\kappa e^{-5u\omega/2}-\kappa^2se^{-u\omega/2}\right\}\right]du$$

$$+\frac{i(\lambda+\mu)\delta}{\kappa+1}\int_0^\infty\left[\frac{\kappa e^{izu}}{\kappa^2s^2-t^2}\left\{(\kappa^2s-\kappa s-t)e^{-u\omega/2}+2t\kappa se^{-3u\omega/2}+\kappa te^{-5u\omega/2}\right\}\right.$$

$$\left.+\frac{(1-\kappa)e^{-izu}}{\kappa^2s^2-t^2}\left\{\kappa te^{u\omega/2}+2t^2e^{-u\omega/2}+\kappa^2se^{-3u\omega/2}\right\}\right]du \qquad (176)$$

Equations (175) and (176) give the expressions for additional complex potential which are superposed on those given by (162) to get the resulting complex

potentials.

Now the additional stress-field is computed by substituting the values of $\phi'(z)$ and $\psi'(z)$ from (175) and (176) in equations (11a) and (11b). After simplification, the process yields the following equations from where the Cartesian components of additional stress can be easily computed :

$$p_{xx}+p_{yy} = P_{xx}+P_{yy} =$$

$$= -\frac{4(\lambda+\mu)\delta}{k+1}\int_0^\infty \frac{u\cos(ux)}{k^2s^2-t^2}\Big[e^{-uy}\Big\{-(ks+t)e^{uc_0/2}+2t^2e^{-uc_0/2}+kte^{uc_0/2}+k^2se^{-3uc_0/2}\Big\}$$

$$-e^{uy}\Big\{(ks+t)e^{-uc_0/2}-2kse^{-3uc_0/2}-kte^{-5uc_0/2}-k^2se^{-uc_0/2}\Big\}\Big]\,du \qquad (177)$$

$$p_{yy}-p_{xx}+2ip_{xy} = P_{yy}-P_{xx}+2iP_{xy}$$

$$= -\frac{2(\lambda+\mu)\delta}{k+1}\int_0^\infty \frac{u\cos(ux)}{k^2s^2-t^2}\Big[(2yu+k)e^{-uy}\Big\{-(ks+t)e^{uc_0/2}+2t^2e^{-uc_0/2}+kte^{uc_0/2}+k^2se^{-3uc_0/2}\Big\}$$

$$-\{2yu+1-k\}e^{uy}\Big\{(ks+t)e^{-uc_0/2}-2kse^{-3uc_0/2}-kte^{-5uc_0/2}-k^2se^{-uc_0/2}\Big\}\Big]\,du$$

$$-\frac{2i(\lambda+\mu)\delta}{k+1}\int_0^\infty \frac{u\sin(ux)}{k^2s^2-t^2}\Big[(2yu+k)e^{-uy}\Big\{-(ks+t)e^{uc_0/2}+2t^2e^{-uc_0/2}+kte^{uc_0/2}+k^2se^{-3uc_0/2}\Big\} \qquad (178)$$

$$+\{2yu+1-k\}e^{uy}\Big\{(ks+t)e^{-uc_0/2}-2kse^{-3uc_0/2}-kte^{-5uc_0/2}-k^2se^{-uc_0/2}\Big\}\Big]\,du$$

The resulting stress-field is obtained by superposing additional stress-field upon that obtained by complex potentials (162). Since the additional stress field and the elastic properties of the inclusion and the strip are the same, there will be perfect bond on the interface. The problem thus theoretically be deemed to be solved. However, the results are still quite complicated and for a given case, the results are to be evaluated numerically. The normal shearing an hoop-stresses are of some interest and are formulated below :

$$p_{RR} = \frac{(\lambda+\mu)\delta}{K+1}\int_0^\infty \frac{u}{K^2s^2-t^2}\Big[e^{-uR\sin\theta_1}\Big\{(2uR\sin\theta_1+uc_0+K)\cos(uR\cos\theta_1+2\theta_1)$$

$$-2\cos(uR\cos\theta_1)\Big\}\Big\{-(Ks+t)+Kt+2t^2e^{-uc_0}+K^2s\,e^{-2uc_0}\Big\}$$

$$-e^{uR\sin\theta_1}\Big\{\cos(uR\cos\theta_1-2\theta_1)\,(2uR\sin\theta_1+uc_0+1-K)-2\cos(Ru\cos\theta_1)\Big\}\times$$

$$\times\Big\{Ks+t-K^2s-2Ks\,e^{-uc_0}-tK\,e^{-2uc_0}\Big\}\Big]\,du \qquad \textbf{(179)}$$

$$p_{\theta_1\theta_1} = -\frac{(\lambda+\mu)\delta}{K+1}\int_0^\infty \frac{u}{K^2s^2-t^2}\Big[e^{-uR\sin\theta_1}\Big\{(2uR\sin\theta_1+uc_0+K)\cos(uR\cos\theta_1+2\theta_1)$$

$$+2\cos(Ru\cos\theta_1)\Big\}\Big\{-(Ks+t)+Kt+2t^2e^{-uc_0}+K^2s\,e^{-2uc_0}\Big\}-$$

$$-e^{uR\sin\theta_1}\left\{(2uR\sin\theta_1+uc_0+1-K)\cos(Ru\cos\theta_1-2\theta_1)+2\cos(Ru\cos\theta_1)\right\}\times$$

$$\times\left\{Ks+t-K^2s-2Kse^{-uc_0}-Kte^{-2uc_0}\right\}\Big]du \tag{180}$$

and

$$p_{R\theta_1}=-\frac{(\lambda+\mu)\delta}{(K+1)}\int_0^\infty\frac{u}{K^2s-t^2}\Big[e^{-uR\sin\theta_1}\left\{(2uR\sin\theta_1+uc_0+K)\sin(uR\cos\theta_1+2\theta_1)\right\}\times$$

$$\times\left\{-(Ks+t)+Kt+2t^2e^{-uc_0}+K^2s\,e^{-2uc_0}\right\}+e^{uR\sin\theta_1}\left\{(2uR\sin\theta_1+uc_0+1-K)\times\right.$$

$$\left.\sin(uR\cos\theta_1-2\theta_1)\right\}\left\{Ks+t-K^2s-2Kse^{-uc_0}-Kte^{-2uc_0}\right\}\Big]du \tag{181}$$

Thus the additional, normal, hoop and shearing stresses on the boundary of the inclusion are evaluated by (179) - (181). These may be superposed on the normal, hoop, and shearing stress, on the interface, given by (163) and (164) to give the expressions for resultant normal, hoop and shearing stresses as follows (on the inclusion of radius unity) :

$$p^{b}_{RR} = P^{b}_{RR}$$

$$= \frac{(\lambda+\mu)\,\delta}{k+1}\int_0^\infty \frac{u}{k^2s^2-t^2}\Big[\,e^{-u\sin\theta_1}\big\{(2u\sin\theta_1+uc_0+k)\cos(u\cos\theta_1+2\theta_1)-2\cos(u\cos\theta_1)\big\}\times$$

$$\times\big\{-(ks+t)+kt+2t^2e^{-uc_0}+k^2se^{-2uc_0}\big\}-e^{u\sin\theta_1}\big\{(2u\sin\theta_1+uc_0+1-k)$$

$$\cos(u\cos\theta_1-2\theta_1)-2\cos(u\cos\theta_1)\big\}\big\{ks+t-k^2s-2kse^{-uc_0}-kte^{-2uc_0}\big\}\Big]du \qquad (182)$$

$$-\frac{2(k-1)(\lambda+\mu)\,\delta}{k+1}$$

$$p^{b}_{\theta_1\theta_1} = P^{b}_{\theta_1\theta_1} - \frac{4(k-1)(\lambda+\mu)\,\delta}{k+1}$$

$$= -\frac{(\lambda+\mu)\delta}{(k+1)}\int_0^\infty \frac{u}{k^2s^2-t^2}\Big[\,e^{-u\sin\theta_1}\big\{(2u\sin\theta_1+uc_0+k)\cos(u\cos\theta_1+2\theta_1)$$

$$+2\cos(u\cos\theta_1)\big\}\big\{-(ks+t)+kt+2t^2e^{-uc_0}+k^2se^{-2uc_0}\big\}$$

$$-e^{u\sin\theta_1}\big\{(2u\sin\theta_1+uc_0+1-k)\cos(u\cos\theta_1-2\theta_1)+2\cos(u\cos\theta_1)\big\} \qquad (183)$$

$$\times\big\{ks+t-k^2s-2kse^{-uc_0}-kte^{-2uc_0}\big\}\Big]du$$

$$-\frac{2(k-1)(\lambda+\mu)\delta}{k+1}$$

$$p^{b}_{R\theta_1} = P^{b}_{R\theta_1}$$

$$= -\frac{(\lambda+\mu)\delta}{K+1}\int_0^\infty \frac{u}{K^2s^2-t^2}\Big[e^{-u\sin\theta_1}\Big\{(2u\sin\theta_1+uC_0+K)\sin(u\cos\theta_1+2\theta_1)\Big\}\times$$

$$\times\Big\{-Ks-t+Kt+2t^2e^{-uC_0}+K^2se^{-2uC_0}\Big\}+e^{u\sin\theta_1}\Big\{(2u\sin\theta_1+uC_0+1-K)\times$$

$$\sin(u\cos\theta_1-2\theta_1)\Big\}\Big\{Ks+t-K^2s-2Kse^{-uC_0}-Kte^{-2uC_0}\Big\}\Big]du \tag{184}$$

In the appendix following this chapter the values of the resultant normal, shearing and hoop stresses for the inclusion are given in form of tables in the manner shown in preceding chapter. Since the normal and shear stresses are the same as for the inclusion. Therefore the values of the resultant hoop stress for the matrix at equilibrium interface are given. As in preceding chapter, ν has been taken to be equal to 1/3 and $C_0=L$ takes the values 3, 4, 5, 6, 7, 8, 9 and 10 where the reasons for choosing such values are given in previous chapter.

Appendix to Chapter XI

TABLE 3

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=3	
0	-2.51709664	-0.27235769
10	-2.41438577	-0.28472007
20	-2.30205205	-0.24831272
30	-2.19605517	-0.14122189
40	-2.13841659	0.06011975
50	-2.22734469	0.35258521
60	-2.49279934	0.65699493
70	-3.00741500	0.72010909
80	-3.56129649	0.56063021
90	-3.80840260	-0.00000000
	L=4	
0	-2.30968314	-0.13722238
10	-2.25304320	-0.14236925
20	-2.19775248	-0.11548709
30	-2.15805286	-0.05369574
40	-2.15376249	0.03946157
50	-2.20621562	0.14621804
60	-2.32501632	0.22967511
70	-2.48874494	0.24283577
80	-2.63728189	0.15813890
90	-2.69796625	-0.00000000
	L=5	
0	-2.20444545	-0.07634768
10	-2.17150092	-0.08012809
20	-2.14079297	-0.06427551
30	-2.12136260	-0.02977700
40	-2.12271664	0.01786008
50	-2.15178487	0.06690729
60	-2.20773926	0.10026476
70	-2.27734444	0.10130658
80	-2.33622432	0.06391224
90	-2.35941976	-0.00000000

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=6	
0	-2.14447212	-0.04623730
10	-2.12394950	-0.04047228
20	-2.10510781	-0.04020778
30	-2.09370488	-0.01991735
40	-2.09454839	0.00706803
50	-2.11121788	0.03344479
60	-2.14087120	0.05024317
70	-2.17604923	0.05098712
80	-2.20480293	0.03119262
90	-2.21592557	-0.00000002
	L=7	
0	-2.10728547	-0.03992969
10	-2.09371850	-0.03276104
20	-2.08125854	-0.02723918
30	-2.07372758	-0.01464098
40	-2.07416898	0.00190583
50	-2.08378240	0.01769696
60	-2.10090849	0.02750964
70	-2.12077844	0.02747333
80	-2.13673082	0.01707072
90	-2.14284185	-0.00000003
	L=8	
0	-2.08272001	-0.02041280
10	-2.07330486	-0.02289373
20	-2.06458315	-0.01954217
30	-2.05920523	-0.01135995
40	-2.05913725	-0.00057627
50	-2.06499249	0.00962730
60	-2.07552043	0.01596379
70	-2.08769519	0.01616562
80	-2.09727454	0.01007487
90	-2.10094762	-0.00000006

θ_1	NORMAL STRESS	TANGENTIAL STRESS
	L=9	
0	-2.06567734	-0.01451395
10	-2.05887982	-0.01668884
20	-2.05249971	-0.01463680
30	-2.04844651	-0.00911364
40	-2.04807311	-0.00175691
50	-2.05173430	0.00522083
60	-2.05851114	0.00961309
70	-2.06631643	0.00997474
80	-2.07290553	0.00626392
90	-2.07485786	-0.00000000
	L=10	
0	-2.03338481	-0.01067427
10	-2.04831511	-0.01258527
20	-2.04348177	-0.01133565
30	-2.04030591	-0.00749711
40	-2.03976557	-0.00228541
50	-2.04209056	0.00289923
60	-2.04660529	0.00590671
70	-2.05184549	0.00635532
80	-2.05600229	0.00403909
90	-2.05758107	-0.00000000
	L=11	
0	-2.04423431	-0.00807221
10	-2.04034969	-0.00975759
20	-2.03658286	-0.00901666
30	-2.03402081	-0.00628398
40	-2.03339875	-0.00247928
50	-2.03488103	0.00120698
60	-2.03796658	0.00364400
70	-2.04159707	0.00413099
80	-2.04448661	0.00267026
90	-2.04558474	-0.00000000

θ1	NORMAL STRESS	TANGENTIAL STRESS
	L=12	
0	-2.03724274	-0.00824260
10	-2.03419775	-0.00774169
20	-2.03119263	-0.00733011
30	-2.02907836	-0.00534680
40	-2.02842540	-0.00249970
50	-2.02936077	0.00030406
60	-2.03150910	0.00221404
70	-2.03408614	0.00273992
80	-2.03614932	0.00179299
90	-2.03693479	-0.00000000
	L=13	
0	-2.03178251	-0.00693373
10	-2.02934894	-0.00636284
20	-2.02690416	-0.00606794
30	-2.02512753	-0.00480624
40	-2.02447322	-0.00263127
50	-2.02504951	-0.00024854
60	-2.02656022	0.00128609
70	-2.02842411	0.00177381
80	-2.02992871	0.00121222
90	-2.03050309	-0.00000000
	L=14	
0	-2.02743796	-0.00396236
10	-2.02546039	-0.00515126
20	-2.02343807	-0.00510045
30	-2.02192289	-0.00401019
40	-2.02128413	-0.00231943
50	-2.02161101	-0.00058655
60	-2.02268651	0.00067184
70	-2.02405393	0.00114201
80	-2.02516967	0.00081764
90	-2.02559716	-0.00000000

TABLE 4

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=3	
0	-2.42150569	1.57849428
10	-2.57524851	1.42475149
20	-2.74791873	1.25208126
30	-2.92854929	1.07145068
40	-3.07954004	0.92045994
50	-3.12500039	0.87499960
60	-2.96962932	1.03037067
70	-2.58538574	1.41461425
80	-2.13219276	1.86780722
90	-1.92354397	2.07645601
	L=4	
0	-2.22758487	1.77241513
10	-2.31328416	1.68671583
20	-2.40040278	1.59959719
30	-2.47568977	1.52431023
40	-2.51959425	1.48040573
50	-2.50978270	1.49021728
60	-2.43376121	1.56623878
70	-2.30791569	1.69208430
80	-2.18595502	1.81404495
90	-2.13489121	1.86510876
	L=5	
0	-2.14222437	1.85777561
10	-2.19213706	1.80786292
20	-2.24074796	1.75925201
30	-2.27915302	1.72084697
40	-2.29763472	1.70236525
50	-2.28852907	1.71147092
60	-2.25132751	1.74867247
70	-2.19738984	1.80261013
80	-2.14902890	1.85097107
90	-2.12955192	1.87044807

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=6	
0	-2.09734568	1.90265431
10	-2.12839569	1.87160434
20	-2.15810809	1.84189190
30	-2.18067601	1.81932996
40	-2.19076896	1.80923103
50	-2.18526546	1.81473449
60	-2.16543959	1.83456640
70	-2.13919179	1.86180818
80	-2.11463703	1.88536297
90	-2.10532737	1.89467262
	L=7	
0	-2.07085574	1.92914425
10	-2.09133178	1.90866822
20	-2.11082986	1.88917011
30	-2.12544379	1.87455618
40	-2.13195962	1.86804037
50	-2.12890247	1.87109752
60	-2.11755776	1.88244221
70	-2.10226142	1.89773856
80	-2.08925846	1.91074152
90	-2.08416802	1.91583197
	L=8	
0	-2.05390835	1.94609162
10	-2.06807625	1.93192373
20	-2.08159500	1.91840497
30	-2.09173441	1.90826558
40	-2.09640217	1.90359782
50	-2.09480579	1.90519422
60	-2.08797047	1.91202952
70	-2.07873505	1.92126493
80	-2.07093316	1.92906681
90	-2.06789237	1.93210761

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=9	
0	-2.04240572	1.95759425
10	-2.05260187	1.94739813
20	-2.06238964	1.93761033
30	-2.06979215	1.9[illegible]782
40	-2.07336292	1.92663707
50	-2.07262498	1.92737499
60	-2.06835240	1.93164758
70	-2.06247983	1.93752013
80	-2.05751491	1.94248508
90	-2.05558217	1.94441792
	L=10	
0	-2.03423783	1.96576215
10	-2.04181689	1.95818309
20	-2.04915398	1.95084599
30	-2.05477291	1.94522707
40	-2.05762506	1.94237491
50	-2.05739407	1.94260597
60	-2.05465718	1.94534292
70	-2.05077366	1.94922191
80	-2.04747957	1.95252041
90	-2.04619396	1.95380603
	L=11	
0	-2.02822721	1.97177276
10	-2.03401488	1.96598510
20	-2.03967339	1.96032660
30	-2.04407105	1.95592894
40	-2.04641849	1.95358150
50	-2.04648790	1.95351209
60	-2.04471087	1.95528911
70	-2.04207683	1.95792314
80	-2.03981230	1.96018769
90	-2.03892726	1.96107273

θ_1	HOOP STRESS INSIDE	HOOP STRESS OUTSIDE
	L=12	
0	-2.02367458	1.97632541
10	-2.02819580	1.97180419
20	-2.03266397	1.96733603
30	-2.03619176	1.96380822
40	-2.03816649	1.96183349
50	-2.03841320	1.96158677
60	-2.03725642	1.96274355
70	-2.03543106	1.96456891
80	-2.03383777	1.96616220
90	-2.03321227	1.96678771
	L=13	
0	-2.02014294	1.97985703
10	-2.02374396	1.97625603
20	-2.02734306	1.97265692
30	-2.03023103	1.96976897
40	-2.03192016	1.96807981
50	-2.03226933	1.96773067
60	-2.03152424	1.96847573
70	-2.03024173	1.96975826
80	-2.02909958	1.97090040
90	-2.02864841	1.97135156
	L=14	
0	-2.01734790	1.98265207
10	-2.02026421	1.97973576
20	-2.02321291	1.97678708
30	-2.02561748	1.97438250
40	-2.02708161	1.97291836
50	-2.02748701	1.97251296
60	-2.02702108	1.97297890
70	-2.02611274	1.97388725
80	-2.02528262	1.97471735
90	-2.02495211	1.97504786

Appendix A

In chapters VII and VIII a few integrals are encountered which may be solved by the method given here.

Let
$$f(x)=\frac{2lx}{(x^2+l^2)^2},$$

then substituting in (iv) of (109) we get

$$F(\omega)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{2lx\,dx}{(x^2+l^2)^2(\omega-x)}$$

ω is affix of the point in $y>0$.

To evaluate this we consider

$$\oint_c \frac{2lz\,dz}{(z^2+l^2)^2(\omega-z)}$$

along a contour c consisting of real line from $-R$ to R and a semi-circle Γ below real axis. Then we let $R\to\infty$ and noting that the integral around the semi-circle Γ vanishes and ω is a pole exterior to c, the contour integral becomes equivalent to $F(\omega)$. The value of this integral after some transformation is given by

$$F(z)=\frac{i}{(z+il)^2}.$$

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